LIFE MEMBERSHIP IN THE MATHEMATICAL ASSOCIATION.

In accordance with the action of the Association at its Rochester meeting, members may obtain life membership in the Association by the payment, at the first of any calendar year, of an amount indicated in the accompanying table. In estimating one's age the birthday anniversary nearest to the first of January when payment is made should be taken.

Age	Fee	Age	Fee	Age	Fee
20	$77.84	40	$65.21	60	$43.54
21	77.40	41	64.33	61	42.28
22	76.94	42	63.44	62	41.02
23	76.47	43	62.52	63	39.76
24	75.97	44	61.57	64	38.49
25	75.46	45	60.60	65	37.22
26	74.93	46	59.61	66	35.96
27	74.38	47	58.59	67	34.69
28	73.81	48	57.55	68	33.44
29	73.22	49	56.48	69	32.20
30	72.60	50	55.40	70	30.96
31	71.97	51	54.29	71	29.74
32	71.31	52	53.16	72	28.54
33	70.63	53	52.01	73	27.35
34	69.93	54	50.85	74	26.18
35	69.20	55	49.66	75	25.04
36	68.45	56	48.47	76	23.92
37	67.68	57	47.25	77	22.82
38	66.88	58	46.03	78	21.76
39	66.05	59	44.78	79	20.72

W. D. CAIRNS, *Secretary-Treasurer.*

THE APRIL MEETING OF THE IOWA SECTION.

The twelfth regular meeting of the Iowa Section of the Mathematical Association of America was held, in conjunction with the thirty-seventh annual meeting of the Iowa Academy of Science, at Cornell College, Mount Vernon, Iowa, on April 27 and 28, 1923. The Chairman of the Section, Professor C. W. Emmons, presided at both the Friday afternoon and Saturday morning sessions.

There were thirty-seven in attendance, including the following twenty-five members of the Association: O. W. Albert, E. W. Chittenden, L. M. Coffin, Marian E. Daniells, R. M. Deming, C. W. Emmons, Fay Farnum, C. Gouwens, R. B. McClenon, F. M. McGaw, J. V. McKelvey, Martha McD. McKelvey, E. E. Moots, E. A. Pattengill, J. F. Reilly, H. L. Rietz, Maria M. Roberts, E. R. Smith, C. W. Strom, John Theobald, J. S. Turner, F. M. Weida, C. W. Wester, W. H. Wilson, Roscoe Woods.

Dinner was enjoyed together Friday evening, followed by a number of brief impromptu speeches. At the business meeting the following officers were elected for 1923–1924: Chairman, F. M. McGAW, Cornell College; Vice-chairman,

J. V. McKELVEY, Iowa State College; Secretary-treasurer, J. F. REILLY, University of Iowa. A committee consisting of Professors McClenon, Gouwens and Weida presented the following resolution:

"Resolved that the Iowa section of the Mathematical Association of America hereby expresses its feeling of deep regret at the death of Mr. T. M. Blakslee of Ames. Mr. Blakslee had been for many years a member of this section, and had rendered excellent service to the cause of mathematics, especially through his contributions to the MONTHLY. His death is a loss to our state and to the nation.

"Resolved further that a copy of this resolution be spread upon the minutes of the Iowa section, and published in the MONTHLY."

The next fall meeting of the Section will be held at Des Moines in November, and the next spring meeting at Ames in April, 1924.

The following papers were presented:

(1) "On the correction of a common error in the calculation of the mean deviation from a given frequency distribution" by Professor H. L. RIETZ;

(2) "On the geodesic in four space" by Professor C. GOUWENS;

(3) "A general expression for the scedastic function for the generalized double frequency distribution" by Professor E. R. SMITH;

(4) "Leibnitz's contribution to the history of complex numbers" by Professor R. B. McCLENON;

(5) "Some curves met with in the conformal representation of integral transcendental functions" by Professor McCLENON;

(6) "The definite integral in a first course in calculus" by Professor J. V. McKELVEY;

(7) "Certain preliminaries to the calculus" by Professor C. W. EMMONS;

(8) "The cochleöide" by Professor ROSCOE WOODS;

(9) "On the theory of wave filters with an application to the theory of acoustic wave filters" by Professor E. W. CHITTENDEN;

(10) "Some functional equations suggested by the mean value theorem" by Professor W. H. WILSON;

(11) "The differentiation of the trigonometric functions" by Professor WILSON;

(12) "What is mathematics?" by Professor J. S. TURNER;

(13) "An application of finite differences" by Professor J. F. REILLY;

(14) "The cycloid and its companion" by Professor E. E. MOOTS.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Professor Rietz dealt with the source and elimination of an error in certain familiar textbook methods for the calculation of the mean deviation when values are grouped into class frequencies. That the error in question is important was shown by examples occurring in statistical practice, where the per cent. error in the mean deviation would vary from 0 to 6.

2. Professor Gouwens showed that, by making use of the fundamental lemma

of the calculus of variations and making one change of independent variable, we have a direct method for obtaining the geodesic in four-space without making use of the δ-process so common in the textbooks on relativity.

3. By means of a general expansion of the double frequency function into a series of Hermitian polynomials, Professor Smith obtained the equation of the scedastic curve. The equation brings out a simple relation between the scedastic and the regressional functions.

4. The pioneer work done by Leibnitz in the study of complex numbers, particularly in connection with the solution of algebraic equations, does not seem to have been sufficiently emphasized by historians of mathematics. In this paper Professor McClenon gave an analysis of this part of Leibnitz's contribution, and brought out the fact that he was a direct precursor of Tschirnhausen, Jean Bernoulli, and Euler in this field.

5. In this paper Professor McClenon mentioned a set of curves that have interesting properties. The equations were obtained by setting simple rational functions of e^{zi} equal to a constant.

6. Professor McKelvey outlined a method of introducing and defining the definite integral by algebraic methods without any use of differentiation or the inverse process. A number of physical and geometrical examples were given which can be set up as the limit of a sum of products, the sum being obtainable by use of formulas for arithmetical or geometrical series or sum of the squares of the integers, etc.

7. Professor Emmons showed the advantage to the beginning student of calculus of a thorough familiarity with the principles of curve tracing. Textbooks generally leave the subject of curve tracing until after the derivative has been introduced. A more opportune time for gaining facility in curve tracing is found to be when the functional notation is introduced. The effect upon the shape, size and nature of a given curve produced by special cases of the homographic transformation may be investigated by the student and the insight thus derived will have its effect upon his later progress.

8. Professor Woods, after making some remarks of an historical nature, discussed the different ways of defining the cochleöide. The relations between the cochleöide and other curves were brought out as well as some of the properties of the curve. To the theorems concerning the tangents to the curve he added some remarks regarding the loci connected with the normals. Finally the applications of the curve were brought out and its connection with a recent problem in physics was given.

9. The theories of electric and acoustic wave filters composed of a number of equal sections in series are based upon linear difference equations of the second order with constant coefficients. If the sections of the filter are unlike the coefficients are variable. After obtaining the necessary and sufficient conditions that a wave of given frequency be transmitted from section to section of a filter without attenuation, Professor Chittenden applied these conditions in the case of filters with alternate sections equal. It was found that, in general, filters of this type transmit two distinct bands of frequencies without attenuation.

10. Professor Wilson investigated certain functional relationships, some of which are special cases of the mean value theorem of the differential calculus while others are generalizations of this theorem and its extensions. The most general continuous functions which satisfy the various equations were found and some of the relationships existing between these functions noted.

11. The belief that students entering the first course in calculus have little or no knowledge (either theoretical or practical) of circular measure and that they have not a working knowledge of proportion leads Professor Wilson to advocate the evaluation of the limit of the quotient of the sine of an angle divided by the angle (as the angle approaches zero) in the form $2\pi/R$, where R is the number of angular units contained in the complete angle about a point. This limit is approached by a comparison of areas one of which, the area of a sector, is derived from its ratio to the area of the entire circle. The general formulæ thereafter found for the derivatives of the trigonometric and inverse trigonometric functions are used in the classroom for most of the semester, emphasis being placed on the simplification introduced by the use of radian measure (defined through the use of $R = 2\pi$). Exercises in graphing, rate of change and expansions in series fix the ideas for the student.

12. In this paper Professor Turner characterized the following subjects: common sense, categorical logic, inductive logic, logical reasoning, philosophy, science. Finally two definitions of mathematics were given, in which certain terms previously explained were employed.

13. In the paper "An interpolation formula for equidistant frequency distributions" by Langman in the *Quarterly Publications of the American Statistical Association* is solved the following problem: Given the frequency over each of the unit sub-intervals of the interval from a to b, to find the frequency over any arbitrary interval. Professor Reilly showed how Langman's result can be obtained much more easily and briefly by applying some principles of the calculus of finite differences.

14. The companion to the cycloid, $x = a\theta$, $y = a(1 - \cos\theta)$, has the same period and amplitude as the cycloid. The two curves are tangent at their highest point, and coincident at their lowest points. Professor Moots showed that, if the generating circle be divided into two parts by a diameter and placed in the position corresponding to $\theta = \pi$, this circle, the cycloid and its companion divide the circumscribing rectangle into eight equal parts each of area $\pi a^2/2$. The rectification of the companion leads to the elliptic integral $E(\varphi, k)$, where $\varphi = 180°$ and $k = 1/\sqrt{2}$, which when evaluated gives the length as $7.6401a$, or approximately $.36a$ less than that of the cycloid.

J. F. Reilly, *Secretary-Treasurer.*

THE MAY MEETING OF THE KENTUCKY SECTION.

The seventh regular meeting of the Kentucky Section of the Mathematical Association of America was held at the Physics Building, University of Kentucky,

Lexington, on Saturday, May 5, 1923. The meeting consisted of two sessions with Dr. Elizabeth LeStourgeon, the chairman, presiding.

The attendance was seventeen, including the following members of the Association: R. V. Blair, P. P. Boyd, J. M. Davis, H. H. Downing, A. R. Fehn, Elizabeth LeStourgeon, C. H. Richardson and G. A. Seubert.

Professor A. R. FEHN, Center College, was elected chairman of the Section, and Mr. G. A. SEUBERT, University of Kentucky, secretary-treasurer. The next meeting will be held at Center College next April.

The following papers were read:

(1) "Anallagmatic curves" by Dean P. P. BOYD;

(2) "Some remarks on statistics" by Professor A. R. FEHN;

(3) "A vectorial treatment of the loci of points of a moving body which are exceptional points of their trajectories" by Professor E. L. REES;

(4) "Unified mathematics for freshmen" by Professor C. H. RICHARDSON;

(5) "A treatment by vector methods of order of contact" by Mr. AUGUSTUS SISK (by invitation);

(6) "Illustrated lecture on astronomy" by Professor H. H. DOWNING.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dean Boyd introduced his subject by recalling the necessary facts concerning circular points at infinity and the foci of higher plane curves. Circular inversion was presented as a specialization of the quadratic transformation. The relations between the inverse, the pedal and the polar reciprocal of a curve were demonstrated. Various special anallagmatic curves, bicircular quartics, especially, were shown. The necessary condition for an anallagmatic curve was derived. The curve was derived as the envelope of a pencil of circles orthogonal to the circle of inversion. The geometric construction of points upon and of the foci of the anallagmatic curve, when the deferent is given, was explained. Finally various theorems concerning circular cubics and bicircular quartics as anallagmatic curves were stated.

3. Professor Rees derived by vector methods the equation of the third-order curve of inflections and the third-order surface locus of points at which the osculating planes of their trajectories are stationary. Methods for obtaining the instantaneous loci of points for which the osculating circles and the osculating spheres of their trajectories are stationary were also outlined. The manner in which these loci are related was pointed out and certain properties of these loci were proved in a simple way by vector methods.

4. Assuming that a course of mathematics for freshmen should prepare the student for a more intelligent analysis, interpretation, and understanding of quantitative phenomena, Professor Richardson stated as his opinion that such a course should introduce the student to the more powerful methods for investigating quantitative phenomena; that in view of the recent applications of mathematics to many fields such as psychology, education, biology, economics, the problems should be selected from a wide range of topics; that, where mathe-

matics is required of freshmen, expert facility in some of the processes should be sacrificed in order that an acquaintance with the many applications may be accomplished. He also expressed the belief that such a course emphasizing utility would stimulate a feeling for the need of a better mathematical training, and retrieve for our field some ground lost during the past decade by reason of misplaced emphasis.

5. Mr. Sisk showed by vector methods that in general the tangent line has contact of the first order, the osculating circle has contact of the second, and the osculating plane has contact of the third order.

6. Professor Downing showed about fifty views of the sun, moon, planets, comets, nebulas, star clusters, and star clouds. The principal features of each slide were pointed out.

C. H. RICHARDSON, *Secretary-Treasurer.*

ORGANIZATION MEETING OF THE MICHIGAN SECTION.

A meeting for the purpose of organizing a Michigan Section of the Association was held at the University of Michigan, Ann Arbor, on March 29, 1923. The Mathematics Section of the Michigan Schoolmasters' Club had arranged to have its program for the session of this day of especial interest to teachers of mathematics in the colleges, this to culminate in the organization of the section. Professor Harold Blair of Kalamazoo Normal presided at the first part of the meeting, and then turned the meeting over to Professor T. H. Hildebrandt. After a brief discussion by Professor W. B. Ford outlining the history and purposes of the Association and the policy of the MONTHLY, the organization was consummated in the drawing up of a petition to the Board of Trustees for recognition as a section and in the adoption of by-laws. Professor L. C. Emmons acted as Secretary for this part of the meeting. The following signed the petition and were declared charter members of the section:[1]

N. H. Anning,
*W. S. Barlow,
J. F. Barnhill,
*G. C. Bartoo,
Harold Blair,
*P. N. Blessing,
J. W. Bradshaw,
*W. M. Coates,
*C. J. Coe,
*C. C. Craig,
S. E. Crowe,
Albertus Darnell,
W. W. Denton,
*J. M. Earl,
L. C. Emmons,
J. P. Everett,
Florence E. Field,
Peter Field,
*R. C. Huffer,
M. F. Johnson,
L. C. Karpinski,
*D. K. Kazarinoff,
A. E. Lampen,
*C. E. Love,
J. L. Markley,
*Jane L. Matteson,
*Selah W. Mullen,
A. L. Nelson,
*Ada A. Norton,
H. L. Olson,
W. H. Pearce,
*V. C. Poor,
Clair Reid,
R. B. Robbins,
*L. J. Rouse,
T. R. Running,

[1] Those whose names are starred were not members of the Association at the time, but have since applied, and by action of the section are to be considered charter members of the section.

S. E. Field,
W. B. Ford,
J. W. Glover,
*A. G. Hall,
T. H. Hildebrandt,
L. A. Hopkins,
J. M. Howie,
*R. H. Schoonover,
E. R. Sleight,
*W. H. Wentworth,
*A. Marie Whelan,
C. B. Williams,
*Orpha E. Worden,
Alexander Ziwet.

The following officers were elected for the ensuing year: Chairman, T. H. HILDEBRANDT; Secretary-Treasurer, J. P. EVERETT; Member of the Executive Committee, E. R. SLEIGHT.

J. P. EVERETT, *Secretary-Treasurer.*

THE MAY MEETING OF THE ILLINOIS SECTION.

The fourth annual meeting of the Illinois Section of the Mathematical Association of America was held at Knox College, Galesburg, Illinois, on May 4 and 5, 1923, in conjunction with the Illinois State Academy of Science. There were three sessions. On Friday afternoon, Chairman Sellew called the meeting to order at 2:20. During the latter part of the session Vice-Chairman Comstock presided. The Friday evening session was a joint meeting of the Academy and the Illinois Section. Prof. Sellew presided at the Saturday morning session.

The attendance was forty-one, including the following eighteen members of the Association:

W. E. Cederberg, C. E. Comstock, M. W. Coultrap, A. R. Crathorne, D. R. Curtiss, C. F. Green, Mabel M. Heren, Mayme I. Logsdon, E. B. Lytle, E. J. Moulton, Mary W. Newson, C. I. Palmer, G. H. Scott, G. T. Sellew, H. E. Slaught, M. G. Smith, E. J. Townsend and Alice Winbigler.

The following officers were elected: C. E. COMSTOCK, Chairman; E. J. MOULTON, Vice-Chairman; G. H. SCOTT, Secretary-Treasurer. By a unanimous vote it was decided that the time and place of the next meeting of the Association will be decided by the executive committee.

The Illinois Section and the State Academy were entertained at a banquet on Friday evening, at Knox College.

After the reading of the minutes of the 1922 meeting, the following papers were presented:

(1) "Similar Perspective Triangles" by Professor F. E. WOOD, Northwestern University (by invitation);

(2) "Differentials, History and Uses" by Professor C. I. PALMER, Armour Institute of Technology;

(3) "Some Problems in Probability" by Professor E. J. MOULTON, Northwestern University;

(4) "Unified Mathematics for Freshmen" by Dr. C. F. GREEN, University of Illinois;

(5) "The Pearson School of Statistics at the University of London" by Professor G. T. SELLEW, Knox College;

Report of Committee "On Some Recent Changes in Mathematical Requirements";

(6) "In the Universities" by Professor E. J. MOULTON, Northwestern University;

(7) "In the Colleges of Illinois" by Professor C. E. COMSTOCK, Bradley Polytechnic Institute;

(8) "In the High Schools" by Professor E. B. LYTLE, University of Illinois.

Abstracts of papers, numbered as in the above list, follow:

1. Professor Wood proved that if two similar triangles are perspective with a finite axis, the circles circumscribing the triangles meet in two real points, one of which is the center of perspectivity, and the other the common Miquel point to each of the two quadrilaterals formed by either triangle and the axis of perspectivity. A special case of the Desargues configuration was obtained, and from that a configuration in circles and points. Certain extensions to equilateral triangles, to congruent triangles, to n-sides and to space were suggested.

2. Professor Palmer made a plea that instructors in the calculus should give more attention to clarifying the meaning of a differential and should make greater use of the idea in teaching the calculus to students who later are to study texts in applied mathematics where the differential is of frequent occurrence. He gave the history of the growth of the differential idea during the eighteenth century, and quoted various texts showing the great confusion of the ideas current at the present time.

3. Professor Moulton reported on some problems in probability which come up in a discussion of the accuracy of grades. The most important problem is described thus: If a paper were graded on many occasions by an instructor, the average of the grades may be called the true grade of the paper on the instructor's standard of grading. Suppose now that an instructor gives grades X_1 and Y_1 to two students, and that X_1 is less than Y_1. What is the probability that for the true grades X and Y the former is as large as the latter? The answer of course depends upon the probable error of a grade and the magnitude of the difference $Y_1 - X_1$. The solution applies to other measurements as well as to grades. An interesting conclusion, based on experimental determinations of the probable error, is that for fairly typical semester grades if one student is given a grade of 80 and another 89, then the probability that the former merits as high a grade as the latter is less than one in a hundred thousand. Most of the paper is found in the *Mathematics Teacher*, vol. 16, pp. 141–149, March, 1923.

4. Dr. Green's paper was a short survey of the field of unified mathematics from the viewpoint of its orientation with regard to other reform movements towards increasing the value and significance of our freshman courses in mathematics. The aims of such courses in meeting the needs both of the students who will continue into further courses and of those who will not can possibly best be gained by more emphasis on insight and understanding of fundamental conceptions and modes of thought, by covering as broad a range of mathematical concepts and processes as feasible and by stimulating "functional thinking"

and analysis. These principles are embodied to the fullest extent in unified mathematics.

Also there was outlined the plan used in an experiment with a few classes at the University of Illinois during the present year together with the results attained.

5. Professor Sellew gave some results of his experience in investigating the opportunities for the study of statistics in Europe. He gave a detailed account of these opportunities in the Department of Applied Statistics (under the direction of Professor Karl Pearson) at University College, University of London.

On account of the world-wide reputation of this great center of inspiration for the study of statistics and the curiosity concerning it, the department is overwhelmed with *visitors* from all parts of the world and has been obliged to curtail the amount of time that can be given to them. But both Professor Pearson personally and his twelve able assistants are most generous of the time and attention they give to all *students* in the three laboratories.

6. As a member of a committee requested to report on recent changes in mathematical requirements, Professor Moulton reported tendencies shown by answers to a questionnaire sent out to a number of typical universities and large colleges. The questions were not restricted strictly to the field assigned. Some conclusions follow: In Liberal Arts Colleges entrance requirements are usually limited to one unit of algebra and one unit of geometry, which is a reduction from the requirements of ten or fifteen years ago, in many cases. In the Eastern and Southern states the amount of mathematics offered for admission by students has remained sensibly constant, but elsewhere there has been an appreciable decrease, solid geometry especially showing a slump. Advanced algebra, including quadratics, is now taught in most of the central and western universities, as a regular course for college credit, but not in most of the eastern and southern institutions. Most of the universities give a course in solid geometry, but seldom is it a required course. About one out of four universities have an absolute mathematics requirement for graduation in excess of entrance requirements; most of the others have a group system of requirements, one group including mathematics. Some institutions give no college credit for any mathematics taken in a preparatory school before graduation, but most universities outside of the eastern states give college credit under some circumstances for trigonometry and college algebra taken in high school. Many have tried methods of separating students on the basis of ability, and for the most part they are well satisfied with results.

7. Professor Comstock stated that fifteen colleges in Illinois (not including University of Illinois, University of Chicago, and Northwestern University) report that, due to recent changes, the entrance requirements in mathematics are one unit in algebra and one unit in geometry. In the last ten years there has been a marked decrease in the percentage of Freshmen presenting one and a half units of both algebra and geometry, as is shown by the figures below:

Years.	Per Cent. Alg.	Per Cent. Geom.
1912	83	55
1917	72	40
1922	56	31

The treatment of these two classes of students varies. In some colleges they are put into different sections of Freshman mathematics, the missing half unit being made up in classes requiring more hours a week. In three colleges the Freshmen are divided into sections according to ability rather than according to the number of units presented for entrance. Three colleges hold diagnostic examinations near the beginning of the Freshman year. Only four of the fifteen require college courses in mathematics of all candidates for graduation. None require solid geometry and only six provide for instruction in that subject.

8. Professor Lytle discussed four changes going on in high schools. (1) There is a strong movement for less emphasis of technique and greater simplification of subject matter, evidenced by magazine discussions, new texts, experimental schools and the National Committee Report. Conservatism, drill-master type of teachers, poor preparation of teachers, manipulative type of examination questions and texts make progress slow. The movement includes better types of drill on the fundamental technique retained. (2) The increasing number of Junior High Schools is bringing about what is probably the best reorganization of elementary mathematics we have ever had because it eliminates some of the long-felt loss of time in the eighth year; this reorganization increases correlation possibilities and gives a more real and vitalized mathematics. (3) A popular approval of General Mathematics is evident, but there seems to be a lack of proper differentiation between "fused" or "unified," "correlated" and "general" courses. (4) A new appreciation of real demonstrative geometry has appeared, due probably to the more careful statements of belief of educators on "transfer" and "training" values. However, extended real demonstrative geometry courses are tending to come later and are more often made elective.

G. H. Scott, *Secretary-Treasurer.*

VECTORIAL TREATMENT OF THE MOTION OF A RIGID BODY IN A PLANE.[1]

By E. L. REES, University of Kentucky.

1. Introduction. Although the simplicity and brevity as well as the elegance of vector methods in the treatment of kinematics have long been recognized, few writers on this subject have made use of vector analysis, especially in works published in English. The advantages of the vector treatment are well illustrated

[1] The subject of this paper in so far as it relates to velocities and accelerations was treated from a somewhat different point of view by Professors Ziwet and Field in the Monthly, 1914, 105–113. The method there used is that of Burali-Forti and Marcolongo. In the present paper the Gibbs' notation and method have been used exclusively.

in the applications to the theory of the motion of a rigid system of points in a plane briefly sketched in this paper.[1]

2. Instantaneous Center. We are to study the motion of a plane (the points of which constitute a rigid system) which glides upon a fixed plane. This motion is completely determined at any instant by the velocity of a point P of the moving plane and the angular velocity of that plane. Let $\rho(t)$ be the position vector of this point and let $w(t)$ be the angular velocity vector; also let $q(t)$ be the vector from the point P to any other point Q of the moving plane. The equation of the trajectory of Q is then $r = \rho + q$ and Q's velocity is $v = \dot{\rho} + \dot{q} = \dot{\rho} + w \times q$. Placing $v = 0$ and assuming $w \neq 0$, we find $q = w^{-1} \times \dot{\rho}$. Hence at any instant there is one and only one point at rest (instantaneous center about which the plane is rotating) and its position vector is

$$\rho + w^{-1} \times \dot{\rho}.$$

3. Centrodes. The locus of the instantaneous center in the fixed plane and its locus in the moving plane are called, respectively, the fixed centrode and the moving centrode. The equation of the fixed centrode is obviously $P = \rho + w^{-1} \times \dot{\rho}$, in which t is the scalar parameter.

We refer the moving centrode to the orthogonal unit vectors i', j' (origin at P) in the moving plane, where $i' = \cos\theta i + \sin\theta j$ and $j' = -\sin\theta i + \cos\theta j$, $\theta(= \int w dt)$ being the angle of orientation with reference to the fixed vectors i and j. To find the equation of the moving centrode we have merely to express the vector $Q = w^{-1} \times \dot{\rho}$ in terms of i' and j'; thus the required equation is

$$Q' = Q \cdot i'i' + Q \cdot j'j' = [w^{-1}\dot{\rho}i']i' + [w^{-1}\dot{\rho}j']j',$$

in which i' and j' of the triple products are to be regarded as the functions of t indicated above.

The velocity of the instantaneous center on the fixed centrode is $\dot{P} = \dot{\rho} + \dot{Q}$, and its velocity on the moving centrode relative to the moving plane is

$$\dot{Q}' = \dot{Q} \cdot i'i' + Q \cdot \dot{j}'j' + Q \cdot i'\dot{i}' + Q \cdot j'\dot{j}' = \dot{Q} + \dot{\rho}.$$

Hence the centrodes are tangent to each other at the instantaneous center and the corresponding centrodal arcs are equal. Whence the following theorem:

The moving centrode rolls without slipping on the fixed centrode.

4. Roulettes. When the centrodes $r = P$, $r' = Q'$ are given, the trajectories of the points of the moving plane are called point roulettes. If the generating point be taken as origin in the moving plane, we see at once that the equation of the roulette is

$$r = P - Q = P - Q' \cdot ii - Q' \cdot jj,$$

where i and j of the last two coefficients are to be expressed in terms of i' and j'.

[1] The reader is referred to such elementary works as those of Gibbs-Wilson and Coffin for certain preliminary definitions and theorems which are omitted here for brevity. We use the notation there explained except that primes are used here to indicate vectors which are expressed in terms of the unit reference vectors i', j' of the moving plane. Vectors not affected by primes are expressed in terms of the unit reference vectors i, j of the fixed plane.

If the generating point were any other point Q' of the moving plane, the equation would be

$$r = P + (q' - Q')\cdot ii + (q' - Q')\cdot jj.$$

For a cusp on the roulette $r = p$ we must have $\dot{r} = \dot{p} = 0$, *i.e.*, the generating point must be an instantaneous center at a cusp. Hence in general only points of the moving centrode generate roulettes with cusps, and the locus of these cusps is the fixed centrode.

The envelope of a line fixed in the moving plane is called a line roulette. Let Q be the point of the enveloping line such that the vector q is normal to the line. The equation of the line is then

$$r\cdot q_1 = (p + q)\cdot q_1 = p\cdot q_1 + q.$$

Differentiate with respect to the parameter t and we get $r\cdot\dot{q}_1 = \dot{p}\cdot q_1 + p\cdot\dot{q}_1$, or $r\cdot w_1 \times q_1 = w^{-1}\dot{p}\cdot q_1 + p\cdot w_1 \times q_1$. Solving for r, we have

$$r = r\cdot q_1 q_1 + r\cdot(w_1\times q_1)w_1\times q_1 = (p\cdot q_1 + q)q_1 + (w^{-1}\dot{p}\cdot q_1 + p\cdot w_1\times q_1)w_1\times q_1.$$

Therefore the equation of the envelope is

$$r = p + q + \dot{p}\cdot q_1 w^{-1}\times q_1.$$

When P is at the instantaneous center, $r = p + q$, which shows that the envelope of the line is the locus of the foot of the perpendicular from the instantaneous center to the line.

To find the equation of a line roulette, given the centrodes (if P be a point of the line), we have simply to replace p in the equation $r = p + \dot{p}\cdot q_1 w^{-1}\times q_1$ by $P - Q$, which gives

$$r = P - Q + (\dot{P} - \dot{Q})\cdot q_1 w^{-1}\times q_1,$$

where $Q = Q'\cdot ii + Q'\cdot jj$.

5. Center of Acceleration. The acceleration of any point Q is

$$\ddot{r} = \ddot{p} + \ddot{q} = \ddot{p} + \dot{w}\times q + w\times\dot{q} = \ddot{p} + \dot{w}\times q - w^2 q.$$

For a point of zero acceleration (acceleration center) we have $\ddot{p} + \dot{w}\times q - w^2 q = 0$. Multiply this equation by $q^{-1}\cdot$ and $q^{-1}\times$, and we obtain respectively $\ddot{p}\cdot q^{-1} = w^2$ and $\ddot{p}\times q^{-1} = \dot{w}$, which give the angular speed and the angular acceleration in terms of the acceleration of any point P of the moving plane and the vector from P to the acceleration center. Multiply the second equation by $\ddot{p}\times$, substitute w^2 for $\ddot{p}\cdot q^{-1}$ and we get $w^2\ddot{p} - \ddot{p}^2 q^{-1} = \ddot{p}\times\dot{w}$. Whence $q^{-1} = w^2\ddot{p}^{-1} + \dot{w}\times\ddot{p}^{-1}$, so that $q = (w^2\ddot{p}^{-1} + \dot{w}\times\ddot{p}^{-1})^{-1} = \ddot{p}^2(w^2 p + \dot{w}\times\ddot{p})^{-1}$, and the center of acceleration is $A = p + \ddot{p}^2(w^2\ddot{p} + \dot{w}\times\ddot{p})^{-1}$.

6. Circle of Inflections. We shall now find the locus of points for which the normal acceleration is zero at a given instant. This is equivalent to the problem of finding the locus of the generating points which are inflection points of their trajectories at the instant considered. The condition to be satisfied by the points is

$$(\dot{p} + q)\times(\ddot{p} + \ddot{q}) = 0.$$

Let P be the instantaneous center at that instant. Then $\dot{p} = 0$ and our condition reduces to

$$(w \times q) \times (\ddot{p} + \dot{w} \times q - w^2 q) = -\ddot{p} \cdot qw + w^2 q^2 w = 0,$$

and therefore $\ddot{p} \cdot q^{-1} = w^2$. From this equation we see that the locus of the tip of q^{-1} is a line perpendicular to the acceleration $\ddot{p}$ of the point P. It follows then that the locus of the tip of q, being the inverse of this line with respect to the instantaneous center, is a circle through the instantaneous center with diameter on $\ddot{p}$. This circle is called the "circle of inflections." If we let k equal the diameter $|\ddot{p}|/w^2$, and φ equal the angle $\widehat{Pq}$, positive in the direction of minimum rotation from $\dot{P}$ to $\ddot{p}$, the equation in the scalar polar form is $q = k \sin \varphi$.

The circle of inflections is tangent to the centrodes at the instantaneous center; for, differentiating $P = p + w^{-1} \times \dot{p}$ we find for the centrodal tangent vector $\dot{P} = \dot{p} + \dot{w}^{-1} \times \dot{p} + w^{-1} \times \ddot{p}$, and since $\dot{p} = 0$ we have $\dot{P} = w^{-1} \times \ddot{p}$, which is perpendicular to the diameter of the circle of inflections. Incidentally we have shown that the acceleration of the point P at the instantaneous center is normal to the centrodes.

Let us now find the locus of the points for which the tangential acceleration is zero. Here the condition is

$$(\dot{p} + \dot{q}) \cdot (\ddot{p} + \ddot{q}) = 0,$$

which, under the assumption that $\dot{p} = 0$, reduces to

$$(w \times q) \cdot (\ddot{p} + \dot{w} \times q - w^2 q) = \ddot{p} \times w \cdot q + q^2 w \cdot \dot{w} = 0,$$

and therefore $w \times \ddot{p} \cdot q^{-1} = w \cdot \dot{w}$. This likewise represents a circle; and this circle passes through the instantaneous center and has a diameter on $w \times \ddot{p}$. Since $w \times \ddot{p}$ is perpendicular to $\ddot{p}$ this circle intersects the circle of inflections orthogonally.

The other intersection of these two circles is the center of acceleration. This of course is obvious, but we may make a formal analytical proof by showing that $q^{-1} = w^2 \ddot{p}^{-1} + \dot{w} \times \ddot{p}^{-1}$ satisfies both equations. In fact the two equations (see under "center of acceleration") from which q was found are identically the vector equations of these two circles.

7. Locus of Points of Undulation. If the generating point is a point of undulation (a point at which the tangent has contact of the third order) of the trajectory it is easy to show by means of Taylor's series for Δr that $\dot{r}$, $\ddot{r}$, $\dddot{r}$ are scalar multiples of each other, *i.e.*, $\dot{r} \times \ddot{r} = \ddot{r} \times \dddot{r} = \dddot{r} \times \dot{r} = 0$.

Obviously at a given instant the generating point, if it be a point of undulation, must be on the circle of inflections. Regarding w as constant (a restriction which does not affect the geometric properties of the trajectories) we find at once that the condition $\dot{r} \times \dddot{r} = 0$ is equivalent to $\ddot{p} \cdot q = 0$, when P is at the instantaneous center. Hence, in general, at a given instant there is one and but one generating point at a point of undulation of its trajectory, and this

point is on the circle of inflections. The position vector of this point (origin at P) is perpendicular to $\ddot{p}$.

We proceed to find the envelope of the circles of inflections. Differentiating the equation of the circle of inflections, $\dot{r} \times \ddot{r} = 0$, we obtain $\dot{r} \times \dddot{r} = 0$, which, as we have just seen, is equivalent to $\ddot{p} \cdot q = 0$, the locus of which is the line that intersects the circle of inflections in the point of undulation and the instantaneous center. We know from the theory of envelopes that the points of intersection of this line and circle are also the points of intersection of consecutive circles of inflections and that their locus as t varies is the two-branch envelope of the circles of inflections. Hence the following theorem:

The envelope of the circles of inflections consists of two branches, one of which is the locus of the instantaneous centers (fixed centrode) and the other the locus of the points of undulation.

8. Circle of Cusps. We shall now find the locus of the cusps of the line roulettes which are the points of tangency of the enveloping lines at a given instant. Differentiating the equation of the roulette and placing $\dot{p} = 0$, we have $\dot{r} = w \times q + \ddot{p} \cdot q_1 w^{-1} \times q_1$. At a cusp $\dot{r} = 0$. Hence $wq + (\ddot{p} \cdot q_1 / w) = 0$, or $q + k \sin \varphi = 0$. Whence the theorem:

At any instant the locus of the cusps of the line roulettes which correspond to the enveloping lines is a circle symmetric to the circle of inflections with respect to the tangent at the instantaneous center.

It is from this property that the circle receives its name "circle of cusps."

Obviously the cuspidal tangents all pass through the opposite extremity of the diameter through the instantaneous center.

9. Formula of Savary. Differentiating $r = p + q$ and multiplying by $\cdot q$ we find $\dot{r} \cdot q = 0$ when P is at the instantaneous center. Hence the center of curvature of the trajectory of Q, for the point Q, is on the line joining Q and the instantaneous center.

The equation of the trajectory may also be written $r = P + R$, where P is the instantaneous center and R the vector from P to Q. Denoting by σ the distance from P to the center of curvature, we have for the equation of the locus of the centers of curvature (evolute of the trajectory of Q)

$$r = P + \sigma R_1.$$

Since the tangent of an evolute is normal to the involute, we may write

$$\left[\frac{d}{dt}(P + \sigma R_1)\right] \times R = 0,$$

which reduces to $\dot{P} \times R + \sigma \dot{R}_1 \times R = 0$, or to

$$\dot{P} \times R + \frac{\sigma}{R} \dot{R} \times R = 0,$$

since $R_1 = R/R$. But $p + q = P + R$ and $P = p + w^{-1} \times \dot{p}$, and therefore $R = q - w^{-1} \times \dot{p}$. When P is at the instantaneous center, $R = q$ and $\dot{R} = w \times q - w^{-1} \times p$. Substituting in the equation above, we get $\dot{P} \times q$

$+ \sigma(- qw + \ddot{p}\cdot q_1 w^{-1}) = 0$. Let v be the speed of the instantaneous center. Our vector equation is then equivalent to the scalar equation $vq \sin \varphi + \sigma qw - \sigma wk \sin \varphi = 0$. By differentiating $\boldsymbol{P} = \boldsymbol{p} + \boldsymbol{w}^{-1} \times \dot{\boldsymbol{p}}$, we easily find

$$v = \frac{|\ddot{\boldsymbol{p}}|}{w} = kw.$$

In view of this relation, if $w \neq 0$, our equation is equivalent to

$$\frac{1}{q} - \frac{1}{\sigma} = \frac{1}{k \sin \varphi}.$$

This is known as the *formula of Savary*.

If $\sigma = \infty$, we get as we should the circle of inflections $q = k \sin \varphi$.

If $q = \infty$, we have $\sigma = - k \sin \varphi$. Hence the locus of the centers of curvature of the trajectories of the points at infinity is the circle of cusps.

If we take the time t equal to the centrodal arc s, the curvatures of the fixed and moving centrodes are respectively

$$\boldsymbol{C}_f = \ddot{\boldsymbol{P}} = \ddot{\boldsymbol{p}} + \ddot{\boldsymbol{Q}}$$

and

$$\boldsymbol{C}_m = \frac{d}{dt}(\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}})' = (\ddot{\boldsymbol{Q}} + \ddot{\boldsymbol{p}})\cdot \boldsymbol{i}'\boldsymbol{i}' + (\ddot{\boldsymbol{Q}} + \ddot{\boldsymbol{p}})\cdot \boldsymbol{j}'\boldsymbol{j}' + (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}) \times \boldsymbol{w}\cdot \boldsymbol{i}'\boldsymbol{i}'$$
$$+ (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}) \times \boldsymbol{w}\cdot \boldsymbol{j}'\boldsymbol{j}' = \ddot{\boldsymbol{Q}} + \ddot{\boldsymbol{p}} + (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}) \times \boldsymbol{w}.$$

Subtracting, $\boldsymbol{C}_m - \boldsymbol{C}_f = (\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}) \times \boldsymbol{w}$. Since $\dot{\boldsymbol{Q}} + \dot{\boldsymbol{p}}$ is now a unit vector $C_m \pm C_f = \pm w$, the sign of the second term being plus or minus according as the curvature vectors have opposite or the same directions. The sign of the second member of the equation must be chosen to accord with that of the first. (It should be remembered that C_m, C_f, and w are essentially positive.) This formula states the more or less obvious fact that the rate of turning of the moving plane with respect to the centrodal arc equals the sum or the difference of the curvatures of the centrodes according as their directions have the opposite or the same sense.

Since here $v = 1$, the relation $v = kw$ gives $w = 1/k$ and the formula just found may be written in the equivalent form

$$\frac{1}{\rho_m} \pm \frac{1}{\rho_f} = \pm \frac{1}{k},$$

in which the ρ's are the radii of curvature of the centrodes.

10. Locus of Points of Contact of Stationary Circles of Curvature. For points whose circles of curvature are momentarily stationary, *i.e.*, have contact of the third order (or higher) with the trajectories of those points, we have the following conditions:

$$(\boldsymbol{K} - \boldsymbol{r})\cdot \dot{} = 0, \qquad (\boldsymbol{K} - \boldsymbol{r})\cdot \ddot{\boldsymbol{r}} = \dot{r}^2, \qquad (\boldsymbol{K} - \boldsymbol{r})\cdot \dddot{\boldsymbol{r}} = 3\dot{\boldsymbol{r}}\cdot \ddot{\boldsymbol{r}},$$

where $\boldsymbol{K}$ is the position vector of the center of the circle of curvature. Eliminating $\boldsymbol{K}$ by solving the first two equations (simultaneously with $(\boldsymbol{K} - \boldsymbol{r})\cdot \boldsymbol{k} = 0$) for

$K - r$ and substituting in the third equation, we get $\dot{r}^2[k\dot{r}\,\ddot{r}] = 3\dot{r}\cdot\ddot{r}[k\dot{r}\,\ddot{r}]$, or $\dot{r}^2\dot{r} \times \dddot{r} = 3\dot{r}\cdot\ddot{r}\,\dot{r} \times \ddot{r}$. Replacing r by $p + q$ and letting P be at the instantaneous center, we obtain after simplifying

$$3[w\dot{p}q](\ddot{p}\cdot q - w^2q^2) + w^2\dddot{p}\cdot qq^2 = 0.$$

Consequently the locus of points of contact of stationary circles of curvature is a third order curve This curve obviously passes through the instantaneous center, and since the degree of the term of lowest degree is the second, the instantaneous center is a double point The equations of the tangents at this point are $w \times \ddot{p}\cdot q = 0$ and $\ddot{p}\cdot q = 0$. Consequently these tangents coincide with the common tangent and normal of the centrodes.

Dividing the above equation by w^2q^4, we get

$$3[w^{-1}\dot{p}q^{-1}](\ddot{p}\cdot q^{-1} - w^2) + \dddot{p}\cdot q^{-1} = 0,$$

which shows that the locus of points of contact of stationary circles of curvature is the inverse of a conic with the instantaneous center as center of inversion. This conic is an equilateral hyperbola whose asymptotes have the directions of the tangents of the cubic curve at the double point; and since the conic passes through the origin, the cubic extends to infinity in the asymptotic direction given by

$$(\dddot{p} - 3w \times \ddot{p})\cdot q = 0.$$

11. Locus of Stationary Centers of Curvature. Let C be the vector from the instantaneous center to the center of a stationary circle of curvature. Savary's formula gives us

$$q^{-1} = \frac{w^2C + \ddot{p}\cdot C_1}{\ddot{p}\cdot C}\,C_1.$$

Substituting this expression for q^{-1} in $3[w^{-1}\dot{p}q^{-1}](\ddot{p}\cdot q^{-1} - w^2) + \dddot{p}\cdot q^{-1} = 0$, we find $3[w^{-1}\dot{p}c^{-1}]\ddot{p}\cdot c^{-1} + \dddot{p}\cdot c^{-1} = 0$. Thus the locus of the instantaneously stationary centers of curvature is a curve of the third order having properties similar to those of the cubic found above. There is a double point at the instantaneous center and the tangents of the two cubics at this point coincide with the common tangent and normal of the centrodes.

NOTE ON THE RELIABILITY OF A TEST, WITH SPECIAL REFERENCE TO THE EXAMINATIONS SET BY THE COLLEGE ENTRANCE BOARD.[1]

By W. L. CRUM, Yale University.

Introduction. The occasion for this study was presented by the appearance of the "Ben Wood Report" [2] in early 1922 and the comment thereon appearing on page 2 of the "Advance sheets" of the Board's *Annual Report* for 1921.

[1] Presented before the American Mathematical Society, 28 October, 1922. Since the presentation of the paper, valuable suggestions have been received from Professors E. L. Dodd and C. N. Haskins.

[2] Ben D. Wood, *The reliability and difficulty of the College Entrance Examination Board examinations in Algebra and in Geometry*, published by the Board, 1922.

Mr. Wood defines [1] reliability as the correlation between the results obtained by administering two forms of the same test in succession to the same group of candidates. No such successive applications of the same test having been made, he states that the custom is to use the correlation between the results for the half of the test consisting of the odd-numbered questions and the results for the other half. To secure the reliability from this half-with-half correlation, he uses Brown's [2] formula to "get the reliability of the whole examination from that of the half of it":

$$r_{11} = 2r_{\frac{1}{2}1/11}/(1 + r_{\frac{1}{2}1/11}) \tag{1}$$

where $r_{\frac{1}{2}1/11}$ is the correlation between the halves and r_{11} is the reliability sought.

The objects of this paper are to call attention to the fact that the use of Brown's formula has not been justified and that the conclusions based upon its use are therefore of doubtful validity, and to examine further the problem of estimating the reliability of a test from the results of a single application of that test.

1. We suppose that two forms of the same test are given in succession to the same group of n candidates, and that the deviations from their respective means of the grades of candidate i on parts A and B and on the whole of each test are:

	Part A.	Part B.	Whole.
Test 1	x_i	y_i	w_i
Test 2	s_i	t_i	z_i

The desired measure of reliability is the correlation between w and z:

$$r_{wz} = \frac{\sum w_i z_i}{n\sigma_w\sigma_z} = \frac{\sigma_x\sigma_s r_{xs} + \sigma_x\sigma_t r_{xt} + \sigma_y\sigma_s r_{ys} + \sigma_y\sigma_t r_{yt}}{\sqrt{\sigma_x^2 + 2\sigma_x\sigma_y r_{xy} + \sigma_y^2}\ \sqrt{\sigma_s^2 + 2\sigma_s\sigma_t r_{st} + \sigma_t^2}}. \tag{2}$$

In this notation, Brown's formula is:

$$R = r_{wz} = 2r_{xy}/(1 + r_{xy}) \tag{3}$$

and we note that one set of conditions [3] which will reduce (2) to (3) is:

$$\begin{gathered} \sigma_x = \sigma_y = \sigma_s = \sigma_t, \\ r_{xy} = r_{xs} = r_{xt} = r_{yt} = r_{ys} = r_{st}. \end{gathered} \tag{4}$$

It is obvious that, if we actually administer only the first test, we can know only σ_x, σ_y, and r_{xy}. Not only do we not know the values of the other coefficients, but we have no way of knowing whether they are related by the equalities in (4). If the Wood Report rested upon the assumption of the equalities in (4), no mention was made of that fact. Moreover, it is not stated whether the two standard deviations which can be calculated, σ_x and σ_y, are equal. Furthermore, the Report does not give sufficient of the original data to enable the reader to compute

[1] *Loc. cit.*, p. 4.
[2] William Brown, *Essentials of Mental Measurement*, Camb. Univ. Press, 1911, p. 101.
[3] These are the conditions suggested by Brown: *loc. cit.*, footnote.

these constants; but, judging from the extreme irregularity in the standard deviations of the grades on individual questions, it is very doubtful if σ_x and σ_y can be equal in this case. In fact, Table I on page 5 of the Report enables us to calculate the following:

Question number	1	2	3	4	5	6	7	8	9	10
Mean grade	8.9	9.3	8.4	8.0	2.9	6.7	7.2	8.9	6.8	2.8
Standard deviation	1.9	2.2	3.3	2.9	3.9	4.4	3.2	2.8	1.1	4.3
Coefficient of variation	.22	.23	.39	.36	1.3	.66	.45	.32	.17	1.5

Now, it may be said that the standard deviations of the two halves may still be equal in spite of the lack of uniformity in the dispersions of the several questions. This is certainly true, but the conditions yielding such equality would be extremely exceptional. In fact, there is a simple algebraic relation connecting σ_x with the standard deviations of the individual questions which constitute x and with the correlations of these individual questions with each other. In order for σ_x and σ_y to be equal in spite of the non-uniform dispersion, we must have so great divergences in the correlations of the individual questions with each other as to lead us to an immediate conclusion that the test must have low reliability.

It is accordingly evident that the use of Brown's formula in this instance is not justified by a conformity to the requirements suggested in (4).

2. The equalities (4) are not, however, necessary conditions in order that (2) may reduce to (3): they are sufficient, but the necessary condition is merely that the right members of (2) and (3) be equal to each other.

It is hardly fruitful to go into an extensive analysis of this necessary condition. If one expresses the various symbols in terms of the x_i, y_i, s_i and t_i, and simplifies the resulting equation of condition, one is led to a relation of the 16th degree. Since, by the very nature of the practical problem under consideration, we know nothing precisely about the s_i and t_i, it is not possible to make exact substitutions leading to a simplification of this condition equation. Moreover, it is suggested that we do not have even approximate knowledge—such knowledge as we should base upon considerations of *a priori* probability—of the s_i and t_i; for, here again the essence of the reliability problem is that we do not know the statistical distribution of the grades in the second test. It seems accurate to say that one can not *demonstrate* that the equation of necessary condition is satisfied without actually giving the second test, and that there are no grounds whatever for *assuming* that it is satisfied.

It is not difficult, on the other hand, to find sufficient conditions which are much less restrictive than those given in (4); and which, therefore, have a greater likelihood of being satisfied in actual practice. Suppose we set g equal to the right member of the exact relation (2), and introduce the following notation:

$$\left.\begin{array}{ccc} \Sigma x_i y_i = p, & \Sigma s_i t_i = (1+a)p, & \Sigma x_i s_i = (1+b)p, \\ \Sigma x_i t_i = (1+c)p, & \Sigma y_i s_i = (1+d)p, & \Sigma y_i t_i = (1+e)p; \\ 4A = b+c+d+e, & \Sigma x_i^2 = h, & \Sigma y_i^2 = (1+k)h, \\ \Sigma s_i^2 = (1+l)h, & \Sigma t_i^2 = (1+m)h, & 2K = l+m, \end{array}\right\} \quad (5)$$

and we have:

$$g = \frac{4p(1+A)}{\sqrt{h^2(4+2k+4K+2kK)+2ph(4+2a+k+2K+ka)+4p^2(1+a)}}, \quad (6)$$

$$R = \frac{2p}{h\sqrt{1+k}+p}, \quad (7)$$

and a set of conditions sufficient to reduce (6) to (7) is:

$$a = A = k = K = 0. \quad (8)$$

Although the conditions (8) are rather severe, they are far less so than those imposed by (4). Two of them, $a = 0$ and $k = 0$, are included among the conditions of (4). The other two, $A = 0$ and $K = 0$, are more likely to be fulfilled than the remaining conditions under (4). Indeed, these two conditions may be stated verbally as: the average of the four product-sums of the halves of the first test with the halves of the second test shall be the product-sum of the halves of the first test with each other; and the average of the standard deviations of the halves of the second test shall equal the standard deviation of a half of the first test.

The examination of various hypotheses will of course lead to other possible sets of sufficient conditions for the reduction of (6) to (7). An interesting example is to assume that k is zero, that p is positive, and that b, c, d, e, l, m are all small. It is then possible to expand the right member of (6) and show that it will reduce to (7), except for terms of second and higher orders, if:

$$2A = K = a. \quad (9)$$

Although the conditions (9) are still less restrictive than (8), their practical significance is less apparent, and it is less easy to see how they might be realized in practice. Indeed, it is doubtful whether we can find a group of sufficient conditions the fulfillment of which we have a better chance of establishing on *a priori* grounds than the set given in (8). If we confess our entire inability to show that the necessary condition mentioned at the beginning of this section is satisfied, we must fall back on reasoning about the properties of our distributions as revealed by the dispersions and inter-correlations; and, for this purpose, it is improbable that we shall find a better set of conditions than (8).

3. The chief practical question is, then, under what circumstances we may expect—from a mere knowledge of the first test—the sufficient conditions we have discussed to be fulfilled. The requirement that k be zero, which enters in each group of conditions suggested above, is one which can actually be observed. In a particular test, we can learn by computation whether k is zero; and, with care and practice, we should be able to construct a test in which k is approximately zero. On the same grounds, we should be able to insure that K is zero approximately.

As for the two remaining conditions in (8), it is suggested that one can scarcely hope to have them satisfied in a test such as the entrance algebra or

any other test the parts of which are specifically designed to measure different intellectual capacities—capacities differing certainly in degree and almost surely, in many practical cases, in kind. The assumption that A reduces to zero in such cases is quite unwarrantable: one would expect a marked lack of uniformity in the four inter-correlations and consequently in the four product-sums. Moreover, even a brief experience with the making of mathematics tests convinces one that the construction of two similar tests covering a range of subjects with a view to having the correlation between the two halves of the one test equal that between the halves of the other—or equal to any previously assigned value whatever—is an extremely difficult feat. We are led to conclude that the condition that a should equal zero is also unlikely to be fulfilled.

The essential point is that we can not hope to establish by *a priori* considerations that such a test, covering a range of capacities, will even approximately fulfill the conditions which we impose. There is indeed a remote possibility that it *may* happen to satisfy those conditions, and a slightly greater possibility that it may satisfy the fundamental necessary condition; but such possibility certainly gives no warrant for assuming that the conditions are met. In other words, it does not seem unfair to say that there can be no justification whatever for using Brown's formula in the study of reliability of a test which covers a range of specific topics. It is on such grounds that the analysis in the Wood Report appears to be unsound; and the conclusions based on such analysis, in particular the conclusion in paragraph 2 on page 14, have not been substantiated.

The case is somewhat different with a test designed to measure a single capacity, such as a section of a standard mental test. Here it does not seem impossible to design a test which shall fulfill the requirements (8). The test really consists of a series of similar observations of the same magnitude: the various questions are the instruments of measurement. Clearly, in the nature of the case, it is desirable to have the individual standard deviations all approximately equal and to have a uniformly high correlation of the individual questions with each other. Care and practice will doubtless enable the examiner to insure to a high degree the realization of this ideal. With such a test, it is not difficult to believe that the conditions (8) might be satisfied at least approximately. All this is, however, robbed of much of its importance, when we remark that such a test is after all but a composite of two similar tests given in immediate succession. If we can make the two halves of such a test have high correlation with each other, we should be able to make an entire new test which should have high correlation with the first. Thus the practical problem of estimating reliability is of relatively little moment in the case of a test intended to measure a single capacity.

For the other case, in which there is a strong practical reason for seeking an estimate of the reliability, we have seen that the use of Brown's formula is open to serious objection. Is there any other way in which we can approximate to the reliability of the test from the evidence yielded by a single application of that test? If we are to proceed by correlating one part of the test with another, we must bear in mind that the selection of these parts can hardly be purely by

chance. The Wood report designates a portion of the test consisting of alternate questions as a random half, but this does not accord with the usual significance of the term *random*. The essential requisite for a random sample, in order that the conclusions of the sampling theory may hold, is that the choice of one object be independent of the other choices. Such a condition can not possibly hold in the case before us, in which we select as a sample the half of a limited group of objects. In addition to the special objections, which apply to the selection of the alternate questions in a test designed as mathematics tests usually are designed, we must raise the general objection that no random sample, in the sense of the theory of sampling, is here possible.

Another available method is to select samples by design: to pick out consciously a sample which we may regard as the best representative of the entire test. It would appear desirable to choose as samples two halves which would be equally good representatives of the whole test, and this condition will be met if the two halves have equal standard deviations. With such a choice, it will be found that the correlation of the whole test with either half is given by

$$\sqrt{(1 + r_{xy})/2} \tag{10}$$

and that the partial correlation of the whole with each half is unity. This does not enable us to proceed to an estimate of reliability, for a knowledge of the correlations of the halves with the whole tells us nothing of the correlations of the halves with the second test. The whole first test can not be taken as a random sample of all the tests given, so far as correlation with the halves is concerned, for the direct dependence of the whole on the halves controls the correlation.

We must conclude then that it is impossible to choose random samples and that, no matter how the samples are chosen, the essential dependence of the whole test upon the halves precludes our using the halves as representatives for the purpose of estimating correlation with a subsequent test. In other words, it seems impossible, on theoretical grounds, to solve the reliability problem: we must conclude that the reliability of a test intended to measure a range of capacities can not be estimated from the results of the single application of that test.

SIMILITUDINOUS AND PSEUDO-SIMILITUDINOUS TRANSFORMATIONS IN A PLANE.

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1. Introduction. The purpose of the present paper is to notice the transformations of certain plane figures in a plane. In a plane two congruent triangles, or figures, are superposable without leaving the plane when their equality is of the same kind, that is, when they are both right- or both left- handedly equal, or as we say are both of the same generation. Otherwise the two figures must be said to be symmetrically equal, or of opposite generation, and one cannot be

moved in the plane to superposition with the other without changing its shape. In like manner two similar figures, in a plane, possess similarity of the same kind or they are symmetrically similar. When we speak of two similar figures, in a plane, we shall mean that they are not symmetrically similar unless otherwise specified. As is well known a figure can be transformed to superposition with any similar figure, in its plane, by movement of its points along continuous path curves, the paths being the members of a family of logarithmic spirals, and in this movement the size of the figure changes continuously while its shape remains invariant.[1] This is the well-known linear transformation. We propose to notice it in a slightly different form showing that there are infinitely many continuous path curves which effect this same transformation.

2. Transformations of Similar Triangles. Let z_1, z_2, z_3 be any three complex numbers, represented by three points in a z-plane. Then any three points w_{12}, w_{23}, w_{31}, which satisfy

$$(z_1 - z_2)w_{12} + (z_2 - z_3)w_{23} + (z_3 - z_1)w_{31} = 0, \tag{1}$$

will be the corners of a triangle similar to $z_1z_2z_3$. This equation can be written in the dual or reciprocal form

$$(w_{12} - w_{31})z_1 + (w_{23} - w_{12})z_2 + (w_{31} - w_{23})z_3 = 0. \tag{2}$$

Either of these leads to

$$(w_{12} - w_{31})/(w_{23} - w_{31}) = (z_3 - z_2)/(z_1 - z_2), \tag{3}$$

which proves the similarity of the triangles.

In any given triangle ABC, whose angles are A, B, C and opposite sides a, b, c, the relation $c = be^{iA} + ae^{-iB}$ exists. If z_1, z_2, z_3 are the corners of any triangle similar to ABC, then by (2)

$$z_3c = z_2be^{iA} + z_1ae^{-iB}. \tag{4}$$

If z_1' z_2' z_3' be any other triangle similar to $z_1z_2z_3$, then

$$z_3'c = z_2'be^{iA} + z_1'ae^{-iB}. \tag{5}$$

Let k_1, k_2 be any two numbers subject to $k_1 + k_2 = 1$. Multiply (4) by k_1 and (5) by k_2 and add the equations. The result shows that the points

$$Z_1 = k_1z_1 + k_2z_1', \qquad Z_2 = k_1z_2 + k_2z_2', \qquad Z_3 = k_1z_3 + k_2z_3'$$

are the corners of a triangle similar to ABC. We notice three forms of transformation.

3. Similitudinous Translation. If k_1, k_2 are real numbers the points Z_1, Z_2, Z_3 divide the segments z_1z_1', z_2z_2', z_3z_3' respectively in the same ratio $\lambda = k_2/k_1$, and as λ varies continuously from 0 to 1, the triangle $Z_1Z_2Z_3$ transforms, changing ze continuously without changing shape, along straight lines from $z_1z_2z_3$ to $z_1'z_2'z_3'$. Such a movement may be called a similitudinous translation transformation. This admits of quite a simple proof by elementary geometry.

[1] See Townsend, *Functions of a Complex Variable*, p. 162.

4. Similitudinous Rotation. If k_1, k_2 are complex numbers and $|k_1| = m$, $|k_2| = n$, then

$$\lambda = k_2/k_1 = (n/m)e^{i\alpha}.$$

Now Z_1, Z_2, Z_3 are the vertices of triangles constructed on z_1z_1', z_2z_2', z_3z_3' as bases respectively, and which are similar to ABC. If α is constant the points Z_1, Z_2, Z_3 lie on similar arcs of circles passing from z_1, z_2, z_3 to z_1', z_2', z_3' and the triangle $Z_1Z_2Z_3$ is transformed continuously from $z_1z_2z_3$ to $z_1'z_2'z_3'$ along the arcs of circles; the points turning through the same angular rotation, the moving triangle changes size continuously but keeps its shape invariant. Such a movement may be called a similitudinous rotation transformation.

5. Arbitrary Path-curve Transformation. When k_1, k_2 are complex numbers, $k_1 + k_2 = 1$ defines a triangle whose vertices in orthogonal Cartesian coördinates are (0, 0), (1, 0) and (x, y); m and n are the radial coördinates of (x, y) with respect to (0, 0) and (1, 0). Let (x, y) describe any curve $F(m, n) = 0$ passing through (0, 0) and (1, 0) whose parametric equations can be written

$$x = \tfrac{1}{2}(1 + m^2 - n^2),$$
$$y = \tfrac{1}{2}\sqrt{2m^2n^2 + 2m^2 + 2n^2 - m^4 - n^4 - 1}.$$

If $\varphi(x)$ is an arbitrary function having $\varphi(0) = 0$, and $f(x)$ any arbitrary function, such a path curve will be

$$y = \varphi(x)[f(x) - f(1)]. \tag{6}$$

As k_1 varies from 0 to 1 the points Z_1, Z_2, Z_3 describe path curves similar to (6), the triangle varying continuously in size, but invariant in shape, from $z_1z_2z_3$ to $z_1'z_2'z_3'$. Such a path curve could be more briefly defined by $w = k_1e^{k_2}$.

6. Generalization for Similar Figures. Any two similar figures can be decomposed into triangles, each of which in the one is similar and similarly situated to a corresponding triangle in the other. The transformation of the one figure into the other is effected by the transformation of any one of these triangles into the corresponding one.

7. Similar Figures in Parallel Planes. If there be two similar figures in two parallel planes and a surface be generated by a straight line which intersects their boundaries in two corresponding points, then all sections by planes parallel to the planes of the two figures are similar figures, as is obvious by their orthogonal projection on one of the planes.

8. We may notice that, in a plane, if n masses $m_1, \cdots, m_n$ are respectively placed at the vertices of n similar triangles $u_1v_1w_1, \cdots, u_nv_nw_n$, then the triangle whose vertices are $\Sigma m_ru_r/\Sigma m_r$, $\Sigma m_rv_r/\Sigma m_r$, $\Sigma m_rw_r/\Sigma m_r$ is also a similar triangle.

9. Elementary Applications. We note a few consequent geometrical problems which may be of interest.

(*a*) The process of section 3 serves to construct the double infinity of triangles, similar to a given triangle, inscribed on three given straight lines in a plane. Or, the quadrilaterals similar to a given quadrilateral inscribed on four given straight lines. More generally, the construction of the double infinity of

polygons of n sides, similar to a given polygon, inscribed on n given straight lines in a plane.

(*b*) The same process establishes the following: Given three skew lines in space, there are two systems of parallel planes any plane of which cuts the three lines in three points which are the corners of a triangle of invariant shape, the one system cuts triangles of one generation, the second those of contrary similarity.

(*c*) The process of section 4 serves to construct the triangles of given shape whose vertices are respectively on three given circles in a plane.

(*d*) If there be any closed boundary in a plane, whose area is A, and a similar boundary be placed anywhere in a parallel plane distant h, the volume of the solid bounded by the two planes and the ruled surface whose generators are the lines joining corresponding points on the two boundaries is

$$V = \tfrac{1}{3}hA(1 + \rho^2 + \rho \cos \alpha),$$

where α is the angular turn and ρ the coefficient of stretch of the similar figure.

10. Pseudo-Similarity. We come now to examine two quadrilaterals which possess a certain relation to each other which we may, for lack of better designation, express by saying they are pseudo-similar. Thus, suppose $ABCD$ is any given quadrilateral and there is another one $A'B'C'D'$ which has the following property: there is a point E' such that $\triangle E'A'B'$ is similar to $\triangle CAB$ and $\triangle E'C'D'$ is similar to $\triangle ACD$. Then there will also be a point F' such that $\triangle F'B'C'$ is similar to $\triangle DBC$ and $\triangle F'D'A'$ is similar to $\triangle BDA$. All such quadrilaterals $A'B'C'D'$ are said to be pseudo-similar to each other. The four triangles mentioned above are not independent of each other, when any two are known the other two are determined and the quadrilateral constructed. That there are infinitely many such quadrilaterals is easily shown, for if z_1, z_2, z_3, z_4 represent any quadrilateral $ABCD$, then the w-numbers satisfying

$$(z_1 - z_2)w_{12} + (z_2 - z_3)w_{23} + \cdots + (z_4 - z_1)w_{41} = 0 \tag{7}$$

are the corners of such a quadrilateral. Equation (7) is therefore the general specification of the group of all pseudo-similar quadrilaterals based on z_1, z_2, z_3, z_4. Moreover we can interchange the z's and w's and rearrange (7), writing it

$$(w_{12} - w_{41})z_1 + (w_{23} - w_{12})z_2 + \cdots + (w_{41} - w_{34})z_4 = 0, \tag{8}$$

so that the relation between the z-figure and the w-figure is a reciprocal one and (8) is also the specification of all pseudo-similar quadrilaterals based on $w_{12}w_{23}w_{34}w_{41}$. The proof by means of (7) of the stated properties is quite simple, thus

$$(z_1 - z_2)w_{12} + (z_2 - z_3)w_{23} + (z_3 - z_1)w' = 0$$

constructs, by (1), a triangle $w_{12}w_{23}w'$ similar to $z_1z_2z_3$.

$$(z_3 - z_4)w_{34} + (z_4 - z_1)w_{41} + (z_1 - z_3)w'' = 0,$$

by (1), constructs a triangle $w_{34}w_{41}w''$ similar to $z_3z_4z_1$. These relations put in (7) give $(z_3 - z_1)w' + (z_1 - z_3)w'' = 0$. Therefore $w' = w''$ (= w_{13} say) iden-

tifies E'. In like manner we construct F' $(= w_{42})$. In particular, if in the quadrilateral $z_1z_2z_3z_4$ we consider the parallelogram whose corners are the midpoints of its sides, this parallelogram is quite obviously pseudo-similar to the w-quadrilateral, its E and F points being the midpoints of the diagonals of $z_1z_2z_3z_4$.

11. Transformations of Pseudo-Similar Figures. Let there be a second quadrilateral $w_{12}'w_{23}'w_{34}'w_{41}'$ pseudo-similar to $w_{12}w_{23}w_{34}w_{41}$, then the transformations of sections 3, 4, 5 are, by the same process carried out there, true for these quadrilaterals, if we change the word similar into pseudo-similar. Any quadrilateral can be transformed into a pseudo-similar parallelogram. Quadrilaterals based on parallelograms, lozenges or on squares have a number of interesting properties, some of these based on squares, called skewsquares, have been published in the MONTHLY (*1923*, 120).

12. Generalization. If $f(z)$ is any analytic function throughout the region of its use, then its mean value between two points z_i, z_j is

$$w_{ij} = \frac{1}{z_i - z_j}\int_{z_j}^{z_i} f(t)dt,$$

and $w_{ii} = f(z_i)$, when $z_j = z_i$. Corresponding to each number pair z_i, z_j, or segment z_iz_j, in the z-plane there is a point w_{ij} in a w-plane, or if one prefers in the z-plane. If we choose n points $z_1, \cdots, z_n$ as the corners of a closed polygon, then

$$\left.\begin{aligned}(z_1 - z_2)w_{12} + (z_2 - z_3)w_{23} + \cdots + (z_n - z_1)w_{n1} = 0,\\ (w_{12} - w_{n1})z_1 + (w_{23} - w_{12})z_2 + \cdots + (w_{n1} - w_{n-1,\ n})z_n = 0.\end{aligned}\right\} \tag{9}$$

For every different form of the function $f(z)$ we derive a set of n numbers w_{ij} satisfying (9) and which may be taken as the corners of a w-polygon. We may speak of all such polygons as forming a group of pseudo-similar polygons and, for the purposes of designation, call the w-points corresponding to the diagonals of the z-polygon the foci. If we take $f(z) \equiv z$, then the midpoints of the sides of the z-polygon will be the corners of a w-polygon pseudo-similar to the w-group. The same relation holds between the z-points satisfying (9) and the w-points, the midpoints of the sides of the w-polygon are the corners of a polygon belonging to the group of pseudo-similar z-polygons. It is also clear that any polygon belonging to a pseudo-similar group can be transformed into any other polygon of that group by the processes in sections 3, 4, 5.

13. If in the equations of section 1 we perform the proper elimination we arrive at the standard form of two similar triangles of the same generation

$$\begin{vmatrix} 1, & z_1, & z_1' \\ 1, & z_2, & z_2' \\ 1, & z_3, & z_3' \end{vmatrix} = 0,$$

thus connecting up with the familiar linear transformation in complex variables. In general, if u, v, w, $\cdots$, t are the n corners of a polygon satisfying (9), and u_r, v_r, w_r, $\cdots$, t_r $(r = 1, 2, \cdots, n-1)$ represent $n-1$ pseudo-similar polygons,

then on elimination from the resultant equations (9),

$$\begin{vmatrix} 1, & u, & u_1, & \cdots, & u_{n-1} \\ 1, & v, & v_1, & \cdots, & v_{n-1} \\ 1, & w, & w_1, & \cdots, & w_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & t, & t_1, & \cdots, & t_{n-1} \end{vmatrix} = 0,$$

which justifies somewhat the term pseudo-similar.

14. We comment on the fact that we are dealing with similarity and homogeneous linear equations and that we are concerned with ratios only. In section 10, when constructing a polygon $ABCD$, we may choose arbitrarily a triangle EAB, then ECD has two degrees of freedom, size and position. If we choose we may make C converge to B as a limit, and when C coincides with B the polygon reduces to a triangle as a special case, but now the point $B \equiv C$ must be counted as two coincident points or as a double point. This being understood, any quadrilateral can be transformed, as in section 3, into a pseudo-similar triangle. It is useless to attempt a like reduction in the case of the triangle. But the degrees of freedom in the construction of the kaleidoscope pattern of a polygon of a pseudo-similar group of n corners permits the transformation to one of $n - 1$ corners one of which however is a double corner. This can be continued until such an n-polygon may be transformed into a triangle having multiple vertices, the sum of their multiplicities being $n - 3$.

15. The following consequences are easily established.

(*a*) If two isosceles triangles have their vertical angles equal to α and this vertex common, their bases are opposite sides of a quadrilateral whose diagonals are equal and intersect at the angle α. The other two sides are the bases of isosceles triangles which have a common vertex and vertex angle $\pi - \alpha$.

(*b*) Reading triangles counter-clockwise in a plane, if ABC, AA_1A_2, BB_1B_2, CC_1C_2 are equilateral, then the midpoints of A_2B_1, B_2C_1, C_2A_1 are the corners of an equilateral triangle.

(*c*) If two pseudo-similar polygons lie respectively in two parallel planes and straight lines be drawn through their corresponding corners, there will be formed a prismoid in space. All planes parallel to the bases cut the surface in pseudo-similar polygons.

(*d*) Areas of closed boundaries being counted in the positive sense of description, let any pseudo-similar polygon satisfying (9) be transformed by section 3; four of its positions being determined by k_1, k_1', k_1'', k_1''' and the respective areas indicated by P, P', P'', P'''. Then

$$\begin{vmatrix} P & 1 & k_1 & k_1^2 \\ P' & 1 & k_1' & k_1'^2 \\ P'' & 1 & k_1'' & k_1''^2 \\ P''' & 1 & k_1''' & k_1'''^2 \end{vmatrix} = 0.$$

THE METHOD OF MOMENTS.[1]

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1. Statement of the problem. One of the simplest problems relating to the fitting of a frequency curve may be stated as follows. A number of measurements have been made of a quantity x, say N measurements in all. These measurements do not all agree, but give the different values $x_1, x_2, \cdots, x_n$, occurring $y_1, y_2, \cdots, y_n$ times respectively, so that $y_1 + y_2 + \cdots + y_n = N$. It is desired to find a polynomial

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p,$$

of given degree p, so that the relation $y = F(x)$ shall give a good approximate representation of the frequencies y_k corresponding to the various values of x.

Since any polynomial in x, other than a constant, becomes very large numerically as x becomes positively or negatively infinite, $F(x)$ can not be expected to give a serviceable representation of a frequency distribution for more than a restricted range of values of x. But it may happen that an approximation is required for only a limited part of a frequency curve, and in such a case a polynomial formula may be entirely satisfactory. So the problem stated has practical importance, in addition to the interest which it possesses as one of the simplest representatives of a general type.

The mathematical discussion of the following paragraphs is of much wider application than the first statement of the problem would suggest. The numbers x_k, instead of being the quantities actually observed, may be representatives of groups of observations which have been thrown together to simplify a computation which would otherwise be excessively laborious. If border-line cases have been regarded as distributed between adjacent groups, some of the group-frequencies y_k may be fractions, instead of integers. More generally still, the reasoning holds without change if the statistical interpretation is abandoned altogether, and $x_1, \cdots, x_n$ are taken to be any n distinct real numbers, while $y_1, \cdots, y_n$ are any n real numbers whatever, distinct or otherwise, positive, negative, or zero.

Let

$$z_k = F(x_k) = a_0 + a_1 x_k + a_2 x_k^2 + \cdots + a_p x_k^p \qquad (k = 1, 2, \cdots, n). \quad (1)$$

The goodness of fit of the curve $y = F(x)$ is to be judged by the extent to which $z_1, \cdots, z_n$ approach coincidence with $y_1, \cdots, y_n$ respectively. The problem remains indefinite until it is specified just what conditions of approximation the numbers z_k must satisfy.

If the degree p is taken equal to $n - 1$, there will be n coefficients in $F(x)$,

[1] Presented to the Minnesota Section of the Association at Northfield, Minn., May 19, 1923. The first two sections of this paper do not require a knowledge of the calculus or of the content of higher courses in mathematics on the part of the reader; the third section, in the nature of an appendix, is of more advanced character.

and they can be determined so as to satisfy exactly the n equations [1] $z_k = y_k$. Usually, however, it will be undesirable, if not impracticable, to calculate so many terms, and some less exacting criterion of approximation must be used.

One suggestion would be to determine the a's so as to minimize the sum of the squares of the errors,

$$(z_1 - y_1)^2 + (z_2 - y_2)^2 + \cdots + (z_n - y_n)^2.$$

Whatever theoretical justification the method of least squares may or may not have in this connection, the suggestion is one of the first that would occur to anybody attacking the problem. From a purely algebraic point of view, it may appear open to the objection that without the use of the calculus it is not clear at first glance how the coefficients are actually to be found in a numerical case.

Another suggestion is that the a's be found by the *method of moments*,[2] that is, that they be made to satisfy the equations

$$\begin{aligned}
z_1 &+ z_2 &&+ \cdots + z_n &&= y_1 &&+ y_2 &&+ \cdots + y_n,\\
z_1x_1 &+ z_2x_2 &&+ \cdots + z_nx_n &&= y_1x_1 &&+ y_2x_2 &&+ \cdots + y_nx_n,\\
z_1x_1^2 &+ z_2x_2^2 &&+ \cdots + z_nx_n^2 &&= y_1x_1^2 &&+ y_2x_2^2 &&+ \cdots + y_nx_n^2,\\
&\cdots\cdots &&\cdots\cdots &&\cdots\cdots &&\cdots\cdots &&\cdots\cdots\\
z_1x_1^p &+ z_2x_2^p &&+ \cdots + z_nx_n^p &&= y_1x_1^p &&+ y_2x_2^p &&+ \cdots + y_nx_n^p.
\end{aligned} \tag{2}$$

If the z's are expressed by means of (1), there will be $p + 1$ equations of the first degree, entirely explicit in form, to determine the $p + 1$ unknowns $a_0, \cdots, a_p$. The first equation requires that the total number of observations, as calculated from the approximation formula, have the true value N. The first two equations together make the arithmetical mean of a set of x's [3] distributed according to the frequencies z_k coincide with that found from the observed frequencies y_k, and the third equation, taken with the first two, means that the calculated and observed standard deviations agree. The subsequent equations are perhaps to be regarded as suggested by analogy. It may be objected that it is not clear just what meaning the later equations have for the closeness of the approximation.

The objections to both suggestions are removed, and their advantages are combined, by the following theorem, which it is the purpose of this paper to prove:

THEOREM: *The method of moments gives the solution of the problem of least squares.*

This important fact, while well known, and indeed immediately apparent to anybody familiar with the ordinary procedure in solving linear equations approximately by the method of least squares, seems not to be so universally recognized as to render superfluous an exposition of it here, with a simple algebraic proof. The method of proof is one of standard utility in discussions of this nature. As

[1] These equations are of the first degree in the unknowns $a_0, a_1, \cdots, a_p$, and their determinant is of a well-known type which is always different from zero; cf. third footnote to section 3 below.

[2] See, for example, W. Palin Elderton, *Frequency-curves and correlation*, London, 1906, Chapter III. In practice, the method of moments is used with certain refinements, which are left out of account here.

[3] This is of course an altogether different thing from the mean of the *frequencies*.

the moment equations are in fact simply the "normal equations" of the problem, given by the ordinary rules, the work may be regarded as an elementary algebraic justification of the normal equations, for the case in hand.

It is to be noticed that the reasoning applies only to the fitting of a *polynomial* approximation curve. The conclusion as stated would not be correct for approximation curves of other forms.

2. Proof of identity. A student interested in the theoretical aspects of the two problems under consideration (the problem of least squares and that of moments) would raise the question whether it is clear in advance that each problem has one and just one solution. It will be shown later, necessarily by somewhat less elementary means, that the answer is in the affirmative.[1] For the present, the fact will be assumed, as something easily believed.

This being granted, let it be assumed that the a's have been determined *so as to solve the problem of least squares.* It is to be proved that the same a's must satisfy the moment equations (2). Suppose, if possible, that this is not true; that there is an exponent r, belonging to the range $0 \leqq r \leqq p$, for which

$$z_1x_1^r + z_2x_2^r + \cdots + z_nx_n^r \neq y_1x_1^r + y_2x_2^r + \cdots + y_nx_n^r.$$

Let

$$\sum_{k=1}^{n} (z_k - y_k)x_k^r = S \neq 0.$$

Let h be a constant, about the determination of which more will be said presently, and let

$$F_1(x) = F(x) - hx^r.$$

Then $F_1(x)$ is a polynomial of exactly the same form as $F(x)$, except that the coefficient a_r is replaced by $a_r - h$. Let $F_1(x_k) = z_k'$. Then

$$\sum_{k=1}^{n} (z_k' - y_k)^2 = \sum_{k=1}^{n} (z_k - hx_k^r - y_k)^2$$

$$= \sum_{k=1}^{n} (z_k - y_k)^2 - 2h \sum_{k=1}^{n} (z_k - y_k)x_k^r + h^2 \sum_{k=1}^{n} x_k^{2r}.$$

If $\sum x_k^{2r}$ is denoted by T,

$$\sum_{k=1}^{n} (z_k' - y_k)^2 = \sum_{k=1}^{n} (z_k - y_k)^2 - h(2S - hT).$$

The constant h has been left undetermined so far; let h now be taken equal to S/T. The division by T is certainly possible, because T is a sum of squares, and hence positive. Then

$$\sum (z_k' - y_k)^2 = \sum (z_k - y_k)^2 - \frac{S^2}{T}.$$

[1] It is understood that $n \geqq 2$, and that $p < n - 1$.

The value of S^2/T being positive, it is found that

$$\sum (z_k' - y_k)^2 < \sum (z_k - y_k)^2.$$

But this is impossible, since the a's were supposed determined in the first place so as to make $\sum(z_k - y_k)^2$ the smallest possible sum of squares. So the theorem is proved.

3. Proof of existence and uniqueness. The assumption stated in the first paragraph of the preceding section may be regarded as made up of four parts, (a) that the problem of least squares has at least one solution,[1] (b) that it can not have more than one solution, (c) that the moment equations have at least one solution, (d) that they can not have more than one solution. Only one of these parts, namely (a), was actually used; the discussion really showed that *if* the problem of least squares has a solution, then the moment equations also have a solution, consisting of the same set of a's. The purpose of this concluding section is to complete the justification of (a), (b), (c), and (d). It will be found necessary here to use something more than the methods of elementary algebra.

Let $G = \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}$. Then[2] $|y_k| \leqq G$ for $k = 1, 2, \cdots, n$. If any one of the numbers $|z_k|$ were greater than $2G$, the corresponding difference $|z_k - y_k|$ would be greater than G, and $\sum(z_k - y_k)^2$ would be greater than G^2. For a specified set of numbers z_k, let the equations

$$a_0 + a_1 x_k + \cdots + a_p x_k^p = z_k, \qquad k = 1, 2, \cdots, p + 1,$$

be regarded as a set of linear equations for determining the a's. The corresponding equations for $k = p + 2, \cdots, n$ may be left out of account. (It will be remembered that $p + 1 < n$.) The determinant of the $p + 1$ equations is different from zero;[3] let it be denoted by D. Let the cofactor of x_k^r in this determinant be represented by X_{kr}. By Cramer's rule,

$$a_r = \frac{1}{D} \sum_{k=1}^{p+1} z_k X_{kr}.$$

If H is the largest of the $(p + 1)^2$ numbers $|X_{kr}|$, if the z's are subjected to the restriction that $|z_k| \leqq 2G$, $k = 1, 2, \cdots, p + 1$, and if the number $2(p + 1) \times GH/|D|$ is denoted by K, then[4] $|a_r| \leqq K$, $r = 0, 1, \cdots, p$.

Since $\sum(z_k - y_k)^2$ is a continuous function of the a's, it follows from a fundamental proposition in the theory of functions of real variables that among all sets of a's *subject to the restriction that* $|a_r| \leqq K$, $r = 0, 1, \cdots, p$, there will be

[1] By a "solution" is meant a set of numbers $a_0, a_1, \cdots, a_p$, satisfying the conditions of the problem; and two solutions $(a_0, \cdots, a_p)$, $(a_0', \cdots, a_p')$ are to be regarded as distinct, unless every a_r is equal to the corresponding a_r'.

[2] The discussion is put in such form as to apply even when the y's are not regarded as statistical frequencies, but are allowed to take on arbitrary values, negative as well as positive.

[3] If the determinant were zero, there would exist a set of a's, not all zero, satisfying the equations

$$a_0 + a_1 x_k + \cdots + a_p x_k^p = 0, \qquad k = 1, 2, \cdots, p + 1,$$

and the polynomial $a_0 + a_1 x + \cdots + a_p x^p$ would have $p + 1$ distinct roots.

[4] For the method of proof, cf., *e.g.*, L. Tonelli, I polinomi d'approssimazione di Tchebychev, *Annali di matematica pura ed applicata*, series 3, vol. 15 (1908), pp. 47–119; pp. 61–62.

at least one set $\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_p$ which reduces the sum of squares to a minimum. Furthermore, this minimum, which may be denoted by g^2, is not greater than G^2, since the sum of squares can be brought down to G^2 simply by taking $a_0 = \cdots = a_p = 0$. On the other hand, if any a_r is taken greater than K in absolute value, it follows from the preceding paragraph that at least one of the (first $p+1$ of the) z's must violate the condition $|z_k| \leqq 2G$, and then $\sum(z_k - y_k)^2$ will be *greater* than G^2. So the value of $\sum(z_k - y_k)^2$ can not be reduced below g^2 by any choice of the a's whatever; that is, the numbers $\bar{a}_0, \cdots, \bar{a}_p$ give at least one solution of the problem of least squares, and (*a*) of the first paragraph of the section is proved. The truth of (*c*) then follows from the work of the preceding section, as has already been pointed out. It remains to consider (*b*) and (*d*).

One aspect of the result obtained so far is, that if the x's are held fast, the moment equations (2) will possess at least one solution for arbitrary values of the numbers y_k. But this is equivalent to saying that if the right-hand members are denoted by $Y_0, \cdots, Y_p$, the equations will possess at least one solution for completely arbitrary values of the Y's. For if the Y's are regarded as given, it will always be possible to set $y_{p+2} = \cdots = y_n = 0$, and then to determine $y_1, \cdots, y_{p+1}$ so as to satisfy the $p+1$ equations

$$y_1 x_1^r + y_2 x_2^r + \cdots + y_{p+1} x_{p+1}^r = Y_r, \qquad r = 0, 1, \cdots, p,$$

the determinant of these equations being the conjugate of one which has already been seen to be different from zero. Consequently the determinant of the equations (2), regarded as a set of linear equations for the unknowns $a_0, \cdots, a_p$, must be different from zero,[1] and the equations will have *just one* solution for any given values of the y's. This establishes (*d*).

Finally, if the problem of least squares had two distinct solutions, each of these would give a solution of the moment equations, by section 2, and therefore (*d*) carries with it the truth of (*b*).

It would not be difficult to prove (*a*) and (*b*) on the one hand, or (*c*) and (*d*) on the other hand, without the intervention of section 2, but the present treatment seems simpler when the work of section 2 has been carried through.

ANOTHER DEFINITION OF AMICABLE NUMBERS AND SOME OF THEIR RELATIONS TO DICKSON'S AMICABLES.[2]

By BENJAMIN FRANKLIN YANNEY, Chicago, Illinois.

1. Professor L. E. Dickson (*1913*, 84–92) defines an amicable triple of numbers to be three numbers such that the sum of the proper divisors of each number equals the sum of the remaining two numbers. In a similar manner he defines

[1] If the determinant were zero, there would be values of the Y's for which the equations could not be solved. Suppose the rank of the matrix of the coefficients were $q < p+1$. Let D_q be a particular non-vanishing q-rowed determinant of the matrix. Let the rth equation be one whose coefficients do not enter into D_q, and let $Y_r = 1$, while all the other Y's are zero. Then the

an amicable k-tuple of numbers, which he expresses in symbols thus:

$$\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k) = n_1 + n_2 + \cdots + n_k, \qquad (A)$$

where $\sigma(n)$ denotes the sum of all the divisors of n. He then proceeds to derive nine sets of amicable triples, closing his paper with a list of ten sets of numbers r, s, t with an appropriate equation in a for each set, from which a value for a relatively prime to each number r, s, t is to be found such that ar, as, at shall be an amicable triple. The last one of the set he himself solves, thus giving a tenth set of amicable triples. He proposes (*1913*, 196, *1919*, 214) Number Theory Problem 191 (erroneously numbered 187), which requires the solution of any one of the remaining nine unsolved equations of the list given in the paper referred to.

The purpose of the present paper is to give another definition of an amicable k-tuple of numbers, and to show some relations existing between amicable numbers as thus defined and amicable numbers as defined by Professor Dickson. An application will be made in an attempt to solve one of the ten equations listed by him. Incidentally also, a still more general definition will be suggested, of which the two compared herein are special cases.

2. Accordingly we shall say that three numbers form an amicable triple if each number equals the sum of the proper divisors of the other two. Thus n_1, n_2, n_3 will form an amicable triple if

$$\begin{aligned} n_1 &= \{\sigma(n_2) - n_2\} + \{\sigma(n_3) - n_3\}, \\ n_2 &= \{\sigma(n_3) - n_3\} + \{\sigma(n_1) - n_1\}, \\ n_3 &= \{\sigma(n_1) - n_1\} + \{\sigma(n_2) - n_2\}; \end{aligned} \qquad (1)$$

or, more briefly, if

$$n_1 + n_2 + n_3 = \sigma(n_3) + \sigma(n_1) = \sigma(n_1) + \sigma(n_2) = \sigma(n_2) + \sigma(n_3). \qquad (2)$$

Similarly, $n_1, n_2, \cdots, n_k$ form an amicable k-tuple if

$$\sum_{i=1}^{k} n_i = \sum_{1} \sigma(n_i) = \sum_{2} \sigma(n_i) = \cdots = \sum_{k} \sigma(n_i), \qquad (B)$$

where the indices under $\sum$ indicate which n is to be omitted in the respective summations.

From (B) we get

$$\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k), \qquad (3)$$

which expresses the same relation between the sums of all the divisors of the respective numbers as in (A). But instead of these being equal to the sum of all

$(q+1)$-rowed determinant of the augmented matrix which contains the minor D_q and the element Y_r is different from zero, the augmented matrix is of rank $q+1$, and the equations are inconsistent.

[2] Professor Dickson, to whom the author acknowledges indebtedness for criticisms and suggestions given during the preparation of this paper, must not in any way be held responsible for any infelicities or inaccuracies that may have found their way into the final selection and arrangement of the material employed. It is only fair to state also that definition (B) had occurred to Professor Dickson when he made his investigations, but that he chose (A) because of its greater simplicity.

the numbers, as in (A), we have, from (B) and (3),

$$\sum_{i=1}^{k} n_i = (k-1)\sigma(n_j), \qquad j = 1, 2, \cdots, \text{or } k; \tag{4}$$

so that (2) becomes

$$n_1 + n_2 + n_3 = 2\sigma(n_j), \qquad j = 1, 2, \text{ or } 3. \tag{5}$$

From (4) it easily follows, as indeed it does from the definition directly, the same as in the case of Dickson's amicables, that if $k = 2$, we have the usual case of two amicable numbers. There is, however, one marked difference between (A) and (B). It is the simplicity with which examples may be furnished of amicable k-tuples of numbers as defined in (B). If we let n be any prime number, then $k\,(= n+1)$ such equal prime numbers constitute a k-tuple of numbers amicable by (B). Thus, we have as amicable sets [1] of numbers the following: 2, 2, 2; 3, 3, 3, 3; 5, 5, 5, 5, 5, 5; and so on for every prime. Of a different type is the amicable triple [1]

$$2^2 \cdot 7 \cdot 11,\ 5 \cdot 7 \cdot 13,\ 7 \cdot 83, \tag{6}$$

which set of numbers fulfills (2). The numbers $2^2 \cdot 11$, $5 \cdot 13$, 83 are the three numbers involved in one of Dickson's problems already referred to, and considered later in this paper.

We may state as follows the problem of which (6) is the solution:

Problem. From the three numbers $2^2 \cdot 11$, $5 \cdot 13$, 83 to derive a triple of numbers amicable by (B).

To solve the problem, we note first that the given numbers satisfy (3). What we desire, then, is a number a, not divisible by 2, 5, 11, 13, or 83, such that $(2^2 \cdot 11 + 5 \cdot 13 + 83)a = 2\sigma(83a)$, or such that $2^3 a = 7\sigma(a)$. This latter equation is evidently satisfied if $a = 7$.

3. It will now be shown how we may derive from an amicable set of numbers fulfilling (B), another set fulfilling (A). From (B) we see that

$$k\left(\sum_{i=1}^{k} n_i\right) = (k-1)\left[\sum_{i=1}^{k} \sigma(n_i)\right]. \tag{7}$$

Now let a be a number such that $an_1, an_2, \cdots, an_k$ form an amicable k-tuple in accordance with (A). Then it follows that

$$k\left(\sum_{i=1}^{k} an_i\right) = \sum_{i=1}^{k} \sigma(an_i). \tag{8}$$

From (7) and (8) we obtain

$$\sum_{i=1}^{k} \sigma(an_i) = a(k-1)\left[\sum_{i=1}^{k} \sigma(n_i)\right], \tag{9}$$

[1] "In 1899, E. B. Escott raised the question of the existence of three or more numbers such that each is equal to the sum of the [aliquot] divisors of the others." Dickson's *History of the Theory of Numbers*, vol. I, p. 50.

which, in view of (3), may be written

$$\sigma(an_i) = a(k-1)\sigma(n_i), \qquad i = 1, 2, \cdots, k. \tag{10}$$

This result is a necessary condition. It is also sufficient. For if (10) is true, we obtain the relation (9) by adding the k equations of (10). Then by means of (7) we obtain (8). But from (10), in view of (3), we get, after dividing out k,

$$\sigma(an_1) = \sigma(an_2) = \cdots = \sigma(an_k) = \sum_{i=1}^{k} an_i, \tag{11}$$

which fulfills the conditions in (A). We have thus completed the proof of the

Theorem. *If $n_1, n_2, \cdots, n_k$ form an amicable k-tuple of numbers as defined in (B), then will $an_1, an_2, \cdots, an_k$ form an amicable k-tuple of numbers as defined in (A), if and only if $\sigma(an_i) = a(k-1)\sigma(n_i)$, $i = 1, 2, \cdots,$ or k.*

There follow the corollaries:

Corollary 1. If a is relatively prime to each n_i, then the necessary and sufficient condition of the theorem may be replaced by $\sigma(a) = a(k-1)$.

For if a is relatively prime to n, $\sigma(an) = \sigma(a)\sigma(n)$.

If, further, $k = 3$, a must be a perfect number. In general, if $k > 2$, a must be a multiply perfect number, the order of multiplicity being $k - 1$.

Corollary 2. If a is relatively prime to each n_i, it can not be a prime number.

For then $a + 1 = a(k-1)$, and, hence, $a(k-2) = 1$, which is absurd.

Corollary 3. If n_1, n_2 are amicable numbers, an_1, an_2 can not be amicable for $a > 1$.

For then surely $\sigma(an_i) > a\sigma(n_i)$, and, hence, $\sigma(an_i) > an_j$, $i, j = 1, 2$, $i \neq j$.

4. We now apply some of the foregoing principles to the solution of the first of the list of ten problems proposed by Professor Dickson.

Problem. Given $2^2 \cdot 11$, $5 \cdot 13$, 83, and the equation $7\sigma(a) = 2^4 a$; to find a value for a, not divisible by 2, 5, 11, 13, or 83, such that $2^2 \cdot 11a$, $5 \cdot 13a$, $83a$ shall form an amicable triple [as defined by Dickson].

As already found in the problem previously considered, (6) is an amicable triple in accordance with (B). Hence, we let $a = 7b$. We have, then, from (10) the equation $\sigma(7b \cdot 83) = 2b\sigma(7 \cdot 83)$, from which we get $\sigma(7b) = 2^4 b$. Of this latter equation, every known perfect number except 28 is a solution. But since every known perfect number is even, while the conditions of the problem require that b shall be an odd number, a solution of the problem seems doubtful. Of course, an odd perfect number not divisible by 7, if such exists, would be a solution provided it met also the condition of non-divisibility stated in the problem. There are, however, other solutions than perfect numbers for b in the foregoing equation, as $3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 19$. But this solution of the equation is not a solution of the problem, since it contains two prohibited factors.

5. As examples of sets of triples related as in (10), are 2, 2, 2 and 120, 120, 120; 2, 2, 2 and 672, 672, 672; or, 2, 2, 2 and $2a$, $2a$, $2a$ such that $2a$ is any of the known multiply perfect numbers of multiplicity two. No example has yet been found in which precisely two of the numbers in each set are equal, or

in which the numbers in each set are all distinct. It may be of interest also to note that it can easily be shown that in none of the cases of amicable triples found by Professor Dickson and listed by him in the article referred to does there exist an amicable triple as defined in (*B*), which is related to Dickson's as in (10). Furthermore, in the cases of at least two of the ten problems proposed by him for solution, viz., the second and fifth, it can be shown that there does not exist for either case a corresponding triple of amicable numbers as defined in (*B*). For example, in problem 2 of the list, the corresponding equation in a, necessary for an amicable triple as defined in (10), is $2.3^2\sigma(a) = 19a$. This necessitates a to have some power of 2 as a factor, while at the same time one of the three numbers given in the fundamental set prohibits 2 from being a divisor of a.

6. A More General Definition of Amicable Numbers. The following is suggested as a more general definition of an amicable k-tuple of numbers, which includes definitions (*A*) and (*B*) as special cases:

A set of k numbers $n_1, n_2, \cdots, n_k$ shall be said to form an amicable k-tuple of numbers if

$$\sum_{i=1}^{k} n_i = \sum \sigma(n_{j1}) = \sum \sigma(n_{j2}) = \cdots = \sum \sigma(n_{js}), \qquad s = {}_kC_r, \qquad (K)$$

where each j ranges over r of the k values of i, s being the number of combinations of k things taken r at a time.

From (*K*) again follows (3). Hence, instead of (*K*) we have the simpler expression

$$\sum_{i=1}^{k} n_i = r\sigma(n_j), \qquad j = 1, 2, \cdots, \text{or } k. \qquad (12)$$

We shall refer to r as designating the multiplicity of the amicable set of k numbers. Thus, the multiplicity of the amicable set as defined in (*A*) is 1; and as defined in (*B*), $k - 1$.

As a simple example of (12), we may take the amicable sextuple 2, 2, 2, 2, 2, 2, which is of multiplicity 4. Of a different type is the amicable quadruple $3 \cdot 7 \cdot 41$, $7 \cdot 11 \cdot 13$, $7 \cdot 11 \cdot 13$, $7 \cdot 167$, of multiplicity 3. If each of the four numbers is multiplied by 2, the resulting numbers form an amicable quadruple of multiplicity 2.

In a manner analogous to that used in proving (10), it can be shown that if the numbers $n_1, n_2, \cdots, n_k$ form an amicable k-tuple of multiplicity r, which is to be not less than one half k, then will $an_1, an_2, \cdots, an_k$ form an amicable k-tuple of numbers of multiplicity $k - r$, if and only if

$$(k - r)\sigma(an_i) = ar\sigma(n_i), \qquad i = 1, 2, \cdots, \text{or } k. \qquad (13)$$

ACTUAL SOLUTION OF SIMULTANEOUS LINEAR NUMERICAL EQUATIONS.

By WILLIAM R. RANSOM, Tufts College.

1. Introduction. The theory of the solution of simultaneous linear equations is well known, and can be very briefly and neatly disposed of, as in the fourth chapter of Bôcher's *Introduction to Higher Algebra*. But an actual method of procedure for any case where the coefficients are numerical has not been described in detail in print so far as the writer is aware. It will be the purpose of this paper to state a set of directions in this connection.

2. Notation. We may employ the notation

$$E_k \equiv a_{k0} + a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n$$

and represent a set of linear equations by

$$E_1 = E_2 = \cdots = E_m = 0.$$

From these equations write the n-columned, m-rowed matrix

$$M \equiv \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{Vmatrix}$$

from which certain determinants are to be chosen by striking out rows and columns. A set of a's (from the equations, not from M) whose second suffix is zero will be referred to as the "absolute column." A set of a's whose second suffix is p will be called the x_p-column.

When $(m - p)$ rows and $(n - p)$ columns of M are struck out, the remaining a's form a determinant, of order p, which will be called Δ_p. The rows and columns of M which are represented in Δ_p correspond to certain E's and x's, and it may be supposed, without loss of generality, that the E's and x's are renumbered so that they may be referred to as $E_1 \cdots E_p$, and $x_1 \cdots x_p$. Let them be thus renumbered.

Begin by finding a Δ_p whose value is different from zero. This can always be done [1] since a single non-vanishing coefficient, a_{11}, is a Δ_1 and may be used if one does not easily discover a non-vanishing determinant of higher order.

3. Directions. If $p = m$, proceed at once to (V) below, and solve.

If $p = n$, change the notation, using one of the non-vanishing minors as Δ_p, and proceed as in (I) below.

(I) If p is less than m or n, border Δ_p by means of terms from the $(p + 1)$st row and x_{p+1} column of M, and call the resulting determinant Δ_{p+1}. From Δ_{p+1} take the cofactors of the elements $a_{1,p+1} \cdots a_{p+1,p+1}$, calling them $A_{1,p+1} \cdots A_{p+1,p+1}$, and note that $A_{p+1,p+1} \equiv \Delta_p \neq 0$. Evaluate Δ_{p+1} by means of the expansion formula

$$\Delta_{p+1} = a_{1,p+1}A_{1,p+1} + \cdots + a_{p+1,p+1}\Delta_p.$$

At this point two cases appear: II*a* and II*b*.

[1] The trivial case in which the a's are all zero is excluded.

(II*a*) If $\Delta_{p+1} \neq 0$: begin again at (I), using this higher order determinant as the Δ_p and noting that it will be the last one of the next set of cofactors. (Continue to go back to (I) in this way until a case of (II*b*) is reached. When $p = m$ a row of zeros, or when $p = n$ a column of zeros, is adjoined to M in order that a case of (II*b*) may be reached.)

(II*b*) If $\Delta_{p+1} = 0$: multiply $E_1, E_2, \cdots E_{p+1}$ by $A_{1,p+1} \cdots A_{p+1,p+1}$, respectively, and add the results. As a check on the work thus far, note that the x_{p+1} column adds up to $\Delta_{p+1}x_{p+1}$, and therefore vanishes, and that the $x_1, \cdots x_p$ columns give zero in each case. It is possible that some other columns may give zero: in fact, at this point three cases appear, III*a*, III*b*, III*c*, as follows.

(III*a*) If every x column gives zero, but the absolute column gives

$$a_{10}A_{1,p+1} + \cdots a_{0,p+1}A_{p+1,p+1} \equiv Q \neq 0,$$

the equations are inconsistent, and in this case the work terminates by stating this conclusion and supporting it by the formula just found

$$A_{1,p+1}E_1 + \cdots + \Delta_p E_{p+1} \equiv Q \neq 0$$

which derives from the $(p + 1)$ equations, $E_1 = \cdots = E_{p+1} = 0$, a result inconsistent with them.

(III*b*) If some x column, say the x_{p+2} column, does not give zero (regardless of what happens to the absolute column): the determinant found by bordering the original Δ_p with coefficients from the x_{p+2} column and the E_{p+1} row will be a non-vanishing determinant of order $(p + 1)$, and having Δ_p as minor, just as in (II*a*), and we go back to I as directed in (II*a*), renumbering the columns.

(III*c*) If each column (including the absolute column) gives zero: the equation $E_{p+1} = 0$ is satisfied by any set of values that satisfies $E_1 = \cdots E_p = 0$. In this case the dependence of E_{p+1} upon the first p equations should be exhibited by recording the formula just found

$$E_{p+1} \equiv (A_{1,p+1}E_1 + \cdots A_{p,p+1}E_p) \div (-\Delta_p).$$

Then proceed to (IV).

(IV) Take each remaining equation and treat it as the E_{p+1} in (I). What will happen will be that either:

III*a*: an inconsistency will be found.

III*b*: a higher order non-vanishing determinant will be found.

III*c*: another formula indicating dependence will be found. (III*c*) is continued until each one of the last $(m - p)$ E's is expressed in terms of the first p of them. Then proceed to (V).

(V) The highest order non-vanishing determinant found in the last (III*b*) is now denoted Δ_p and the corresponding p equations are multiplied each by the cofactor of the corresponding element in the last column of Δ_p, and then the equations are added and the result divided by Δ_p. In this addition, the columns containing $x_1 \cdots x_{p-1}$ give zeros, and the result gives x_p, possibly in terms of $x_{p+1} \cdots x_n$ if $n > p$, or as an absolute number if $n = p$.[1]

[1] Since $\Delta_p \neq 0$, n cannot be less than p.

(VI) Take from the Δ_p just used the cofactor of $a_{p,p}$ as a new Δ_p of order lower by one. Substitute the value of x_p just found in (V) into the first $(p-1)$ equations, and proceeding as in (V), find the value of another x, and so on until the value of x_1 is reached.

4. Conclusion. The complete solution will consist of either:

A: a formula showing that certain equations are inconsistent, (III*a*),[1] or

B: A set of formulas for $x_1 \cdots x_p$ found from p of the equations (V), together with $(m-p)$ formulas, if $m > p$, showing how the last $(m-p)$ equations depend upon those which precede them, (III*c*).

Although these directions give no clue as to the theory on which they are based, one who follows them carefully will find that the process explains and justifies itself as it is carried out.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

I. THE TEN MOST IMPORTANT MATHEMATICAL BOOKS IN THE WORLD.

By WALTER C. EELLS, Whitman College, Walla Walla, Wash.

This title is suggested by H. G. Wells's article *The Ten Most Important Books in the World* in the *American Magazine* for April, 1923. "What are the ten most important books in the world?" an interviewer asked Mr. Wells, and in his reply he says, "Absurd questions sometimes make the most interesting discussions. . . . Following the precedents, I will first show how unreasonable a question it is, and then give myself up to its insidious fascination."

The question, "*What are the ten most important mathematical books in the world?*", is equally unreasonable, but to the mathematician it may prove equally or even more fascinating. I suggested a similar question to my class in the History of Mathematics last term, with rather interesting results. I suggest this question now for discussion on the part of the readers of the MONTHLY who may be tempted to yield to its "insidious fascination." To be sure, no two lists will agree, but this very fact will give the discussion its interest and value. What books, if any, will be common to all lists suggested?[2]

The result of such a discussion should be similar to that stated by Mr. Wells in discussing the answer to the question of the six greatest men in the world,

[1] The geometrical and algebraic problems in their conventional form are distinguished at this stage. For most algebraic purposes a system which is inconsistent may be dismissed as such, while the study of a system of linear spreads does not terminate when two of the spreads are found to be parallel.

[2] It is hoped that a number of readers will follow up the suggestion made by the author of this discussion, either by contributing short papers, as in this instance, or by supplying lists of ten which may be of use when the time comes for a final summing up.—EDITOR.

as a result of which he says "endless people were set thinking, very profitably, and sent to their encyclopedias and histories and biographies for refreshing and stimulating reading."

A list of the ten most important mathematical books is not necessarily synonymous with a list of the ten greatest mathematicians. Archimedes or Leibnitz, for instance, should doubtless be included in the latter class, but it is difficult to pick out a single outstanding work of either which nearly approaches the importance and influence of Euclid's *Elements*, or Newton's *Principia*. Much, too, of the important and influential work in mathematics of the modern period has appeared in scattered articles in the journals, not in books.

Neither is a list of the ten most important mathematical books necessarily the ten most important ones for present-day study, any more than is Mr. Wells's list suitable for a similar purpose. In fact less than half of his list does he recommend as valuable reading at the present time.

It is interesting to note that four of Mr. Wells's ten books are scientific, but none of them are mathematical. The nearest he comes is when he considers Newton's *Principia*, "which brought the whole material universe under the domain of natural law," but reluctantly he rejects it as one of his ten.

As a first approximation toward a mathematical list, and as a basis for discussion and suggestion of other lists, I venture to offer the following as my choice, arranged in chronological order, accompanied, in some cases, by noteworthy characterizations of the contents of these books or of their influence which have been made by others.

EUCLID'S "*Elements*" (Alexandria, *c.* 325 B.C.), which "has been for nearly twenty-two centuries the encouragement and guide of scientific thought" (Clifford), which has passed through more than two thousand editions and has exercised such profound influence on the teaching and knowledge of geometry for more than two thousand years, and which is "still regarded by some as the best introduction to the mathematical sciences" (Cajori).

APOLLONIUS'S "*Conic Sections*" (Alexandria ? *c.* 210 B.C.), the great systematic treatise which developed the geometrical "theory of conic sections, and was the prelude to the theory of geometrical curves of all degrees—and of the geometry of form and position" (Cajori) as distinguished from the geometry of measurement.

LEONARDO OF PISA'S "*Liber Abaci*" (Pisa, 1202), which marked the first renaissance of mathematics on Christian soil, introduced Arabian algebra, and brought into general use in Europe the labor-saving Hindu-Arabic numerals, and for centuries was a storehouse of material for later writers on arithmetic and algebra; among others forming the basis for the first printed work on arithmetic, algebra, and geometry, Pacioli's, which was printed at Venice in 1494.

NAPIER'S "*Mirifici Logarithmorum Canonis Descriptio*" (Edinburgh, 1614), which gave to the world Napier's great invention of logarithms with their miraculous power in modern computation, than which "with the exception

of the *Principia* of Newton there is no mathematical work published in the country which has produced such important consequences" (Glaisher, in *Encyclopædia Britannica*).

DESCARTES'S "*Geometrie*" (Leyden, 1637), which in spite of its obscure style was of epoch-making importance in giving to the world the powerful method of analytic geometry "which far transcended everything that ever could have been reached upon the path pursued by the ancients" (Hankel), and than which "there cannot be a language more universal and more simple, . . . and better adapted to express the invariable relations of nature" (Fourier), and which contains in addition the modern exponential and literal notation of algebra.

NEWTON'S "*Principia*" (Full Title: *Philosophiæ Naturalis Principia Mathematica*) (London, 1687), which established the mathematical foundation of the universe, "the greatest production of the human mind" (Lagrange); "the brightest page in the records of human wisdom—and preëminent above all the productions of human intellect" (Brewster's Life of Newton); and which, Laplace says, will always be assured "a preëminence above all the other productions of human genius."

LAGRANGE'S "*Mécanique Analytique*" (Paris, 1788), "an epoch-making work . . . a most consummate example of analytic generality" (Cajori), "a kind of scientific poem" (Hamilton), the foundation of all later work on analytic mechanics, in which Lagrange "impressed on mechanics, as a branch of pure mathematics, that generality and completeness toward which his labours invariably tended" (Ball).

LAPLACE'S "*Mécanique Céleste*" (Paris, 5 vols., 1799–1825), "the translation of the *Principia* into the language of the differential calculus" (Ball), which according to the author was intended to "offer a complete solution of the great mechanical problem presented by the solar system."

BOLYAI'S "*Science Absolute of Space*" (Hungary, 1833), which, although only the appendix to a two-volume work by his father, is characterized by Halsted as "the most extraordinary two dozen pages in the history of human thought," and which, together with Lobachevski's work, opened up the whole fascinating and broadening field of non-Euclidean geometries.

HAMILTON'S "*Lectures on Quaternions*" (Dublin, 1852), "the great discovery of our nineteenth century . . . (in which) there is as much real promise of benefit to mankind as in any event of Victoria's reign" (Thomas Hill), which is the foundation of all modern developments in the field of vector analysis, with its important applications in mathematical physics, including electromagnetic theory and Einstein's generalizations.

It is with much regret that the arbitrary limit of ten forbids the inclusion of such works as Diophantus's "*Arithmetic*," Alkowarezmi's "*Algebra*," Cardan's "*Ars Magna*," Euler's "*Analysin Infinitorum*," Legendre's "*Fonctions elliptiques*" and "*Théorie des Nombres*," Gauss's "*Disquisitiones Arithmeticæ*," Cantor's "*Geschichte der Mathematik*" and others which could easily be mentioned.

This list of ten important books is well distributed among the great fields of mathematics, as well as in time and in nationality. It ranges over twenty-two centuries. In it are represented two Greeks, two (or one) Italians (depending upon whether Lagrange is considered Italian or French), one Scotchman, two (or three) Frenchmen, an Englishman, a Hungarian and an Irishman—a very cosmopolitan group.

II. Note on the Algebraic Solution of the Cubic.

By E. J. Oglesby, New York University.

Take the general cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \tag{1}$$

and assume that the roots x_1, x_2, x_3 have the form [1]

$$x_1 = a + b + c,$$
$$x_2 = a + \omega b + \omega^2 c,$$
$$x_3 = a + \omega^2 b + \omega c,$$

where $\omega^2 + \omega + 1 = 0$ and a, b, and c are to be determined.

We then have the following identities:

$$(a + b + c) + (a + \omega b + \omega^2 c) + (a + \omega^2 b + \omega c) = 3a, \tag{2}$$

$$\begin{aligned}(a + b + c)(a + \omega b + \omega^2 c) + (a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c) \\ + (a + \omega^2 b + \omega c)(a + b + c) = 3(a^2 - bc),\end{aligned} \tag{3}$$

$$(a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c) = a^3 + b^3 + c^3 - 3abc. \tag{4}$$

From (2), (3), and (4)

$$x_1 + x_2 + x_3 = 3a,$$
$$x_1x_2 + x_2x_3 + x_3x_1 = 3(a^2 - bc),$$
$$x_1x_2x_3 = a^3 + b^3 + c^3 - 3abc.$$

From (1)

$$x_1 + x_2 + x_3 = -\frac{3a_1}{a_0},$$

$$x_1x_2 + x_2x_3 + x_3x_1 = \frac{3a_2}{a_0},$$

$$x_1x_2x_3 = -\frac{a_3}{a_0};$$

[1] This assumption may be justified in advance, since a, b, and c can always be found to satisfy these equations, for any x_1, x_2, x_3.—Editor.

whence,

$$a = -\frac{a_1}{a_0}, \tag{5}$$

$$a^2 - bc = \frac{a_2}{a_0}, \tag{6}$$

$$a^3 + b^3 + c^3 - 3abc = -\frac{a_3}{a_0}. \tag{7}$$

From (5) and (6) we get

$$bc = \frac{a_1^2 - a_0 a_2}{a_0^2} \tag{8}$$

and substituting the values of a and of bc into (7), we have

$$b^3 + c^3 = \frac{3a_0 a_1 a_2 - a_0^2 a_3 - 2a_1^3}{a_0^3}. \tag{9}$$

Put $a_0 a_2 - a_1^2 = H$ and $a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 = G$. Then $bc = -(H/a_0^2)$ and $b^3 + c^3 = (-G/a_0^3)$, from which we get, by the solution of a quadratic,

$$b^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2a_0^3}, \qquad c^3 = \frac{-G - \sqrt{G^2 + 4H^3}}{2a_0^3};$$

whence the roots of the cubic are

$$x_1 = \frac{1}{a_0}\left[-a_1 + \sqrt[3]{\frac{-G + \sqrt{G^2 + 4H^3}}{2}} + \sqrt[3]{\frac{-G - \sqrt{G^2 + 4H^3}}{2}}\right],$$

$$x_2 = \frac{1}{a_0}\left[-a_1 + \omega\sqrt[3]{\frac{-G + \sqrt{G^2 + 4H^3}}{2}} + \omega^2\sqrt[3]{\frac{-G - \sqrt{G^2 + 4H^3}}{2}}\right],$$

$$x_3 = \frac{1}{a_0}\left[-a_1 + \omega^2\sqrt[3]{\frac{-G + \sqrt{G^2 + 4H^3}}{2}} + \omega\sqrt[3]{\frac{-G - \sqrt{G^2 + 4H^3}}{2}}\right],$$

It should be noted that this is substantially the same as Cardan's solution but from an entirely different method of attack.

As an example, consider the equation

$$x^3 + 2x^2 + 3x + 4 = 0.$$

Here

$$a = -\frac{a_1}{a_0} = -\frac{2}{3},$$

$$a^2 - bc = \frac{a_2}{a_0} = 1,$$

whence

$$bc = -\frac{5}{9};$$

$$a^3 + b^3 + c^3 - 3abc = -\frac{a_3}{a_0} = -4,$$

whence

$$b^3 + c^3 = -\frac{70}{27}.$$

Therefore

$$b^3 - \frac{125}{729b^3} = -\frac{70}{27},$$

so that

$$b^3 = \frac{-35 + 15\sqrt{6}}{27}, \qquad c^3 = \frac{-35 - 15\sqrt{6}}{27},$$

$$b = -\tfrac{1}{3}\sqrt[3]{35 - 15\sqrt{6}}, \qquad c = -\tfrac{1}{3}\sqrt[3]{35 + 15\sqrt{6}},$$

whence

$$x_1 = a + b + c = -\tfrac{1}{3}[2 + \sqrt[3]{35 - 15\sqrt{6}} + \sqrt[3]{35 + 15\sqrt{6}}],$$

$$x_2 = a + \omega b + \omega^2 c = -\tfrac{1}{3}[2 + \omega\sqrt[3]{35 - 15\sqrt{6}} + \omega^2\sqrt[3]{35 + 15\sqrt{6}}],$$

$$x_3 = a + \omega^2 b + \omega c = -\tfrac{1}{3}[2 + \omega^2\sqrt[3]{35 - 15\sqrt{6}} + \omega\sqrt[3]{35 + 15\sqrt{6}}].$$

III. The "Limiting Case" in Enumerative Geometry.

By Albert A. Bennett, University of Texas.

H. G. Zeuthen, in his work on Enumerative Geometry, entitled *Lehrbuch der Abzählenden Methoden der Geometrie*, points out many significant dangers in enumerative methods. One of the most interesting cautions urged upon the reader is to approach all *special cases* as *limiting cases*. He remarks that in making the count of geometrical constants, usually evidenced in the order of the algebraic equation involved, there is a temptation to treat the special cases by special methods which may introduce new constants or omit, implicitly, significant elements of the general problem which become trivial in the special case. It often happens that a special case is contained in each of two general classes of the same number of parameters. This special case is special, but in a different way for the two classes, and the count must be made differently from the two points of view. As a precaution Zeuthen urges that one treat all such problems as *limiting* cases. Thus in the trivial case of the intersections of a straight line with a curve, a tangent line is to be regarded as the limit of a secant.

When algebraic data are given and algebraic results are desired, is it too much to ask that algebraic methods be employed? Now passing to a limit is *not*

an algebraic method, if one restricts this term severely. One has merely to examine modular geometries to find examples in which secants and tangents exist, for which no limiting processes may be applied. One may indeed replace x by *zero*, but not by letting x *approach* zero. Enumerative results hold in finite geometries with suitable restrictions, but Zeuthen's methods do not apply. What shall we do?

RECENT PUBLICATIONS.

REVIEWS.

Theorie der Gruppen von endlicher Ordnung. By A. SPEISER. Berlin, Julius Springer. 1923. 8vo. viii + 194 pages.

About two years ago there appeared a small volume of 120 pages in the Sammlung Göschen, entitled *Gruppentheorie*, by Ludwig Baumgartner. The fact that the present volume on the same general subject appeared so soon thereafter seems to indicate that there is now a considerable demand in Germany for new introductory books along this line. This is the more remarkable since the German student was already supplied with various useful expository works, including the two books by Netto, published in 1882 and 1908, and Weber's classic *Lehrbuch der Algebra*. While the present volume has much in common with these earlier works its main object seems to be to lead the reader rapidly into some of the more modern developments, especially into those inaugurated by Frobenius about 1896.

The book is divided into 15 chapters, the longest of which is devoted to group characteristics and covers only 20 pages. The applications to crystallography are especially emphasized in view of the simplicity and the precision of these applications. Hence Chapter 6 is devoted to the groups of crystallography, the subheadings being plane lattices, space lattices, and classes of crystals. The first five chapters are devoted mainly to the fundamental theorems relating to finite abstract groups. The reference near the bottom of page 2 should be to volume 1 of Kronecker's *Werke*, not to volume 2, and on page 38 the name L. C. Dickson appears instead of L. E. Dickson.

On page 17 the commutator of A and B is defined as $B^{-1}A^{-1}BA$ instead of $A^{-1}B^{-1}AB$ as is done by A. Loewy on page 189 of volume 1 of the second edition of Pascal's *Repertorium der höheren Mathematik*, 1910, and by H. Vogt on page 540 of tome 1, volume 1, *Encyclopédie des Sciences Mathématiques*. Our author's definition is, however, in accord with the one given by R. Dedekind, who introduced this term, *Mathematische Annalen*, volume 48 (1897), page 553. Although this is not a very important matter it would clearly be desirable to secure uniformity of usage so as to smooth the path of the student who desires to enter this field of study. The author of the present work does not seem to have adopted the custom followed by the majority in this respect. Similar remarks apply to the notation used sometimes to represent the type of a prime power group. On

page 38, for instance, it is noted that the group of isomorphisms of the cyclic group of order 2^m is the abelian group of type $(2^{m-2}, 2)$. This group is more commonly said to be of type $(m - 2, 1)$.

The brevity of the work made it necessary to select the theorems from a large body of known results. Hence our author found it desirable to say only little as regards different possible sets of independent generators of an abstract group. In particular, the theorem that the number of the independent generators of every prime power group is fixed by the group itself and is not dependent upon how the generating operators are selected is not treated. In the small amount of space at his command to lay the foundation for a study of linear substitution groups the author did, however, succeed in giving in a clear and attractive manner a considerable number of the most important theorems relating to abstract groups and to the so-called permutation groups. Chapter 7 is explicitly devoted to the latter.

The proofs of theorems 48 and 49 are vitiated by a slight mistake in the first line of page 45, and the closing developments of Chapter 5 are marred by the use of r in place of s and vice versa, as well as by several statements which are likely to be misunderstood at first by the beginner. For instance, near the middle of page 51 the author states that it is now easy to determine all the p-groups with a cyclic invariant subgroup of index p or of index p^2. In the consideration of the groups of index p it is not assumed that this cyclic invariant subgroup involves an operator of highest order but in the following paragraph this assumption is implicitly made since otherwise the number of the groups coming under the first case would be 6 instead of 3 as here stated. These slight oversights are, however, easily detected by the careful reader and hence are not apt to cause him much annoyance.

Chapters 8 and 9 deal with automorphisms and monomial groups respectively. In the former of these the concepts of automorphism and group of automorphisms are clearly explained and illustrated. This is followed by the ordinary definition of characteristic subgroups to which attention was called recently in this MONTHLY, volume 29, page 320. The automorphisms of abelian groups are treated more fully than those of other groups, and at the close of the section on this subject reference is made to a practical method of establishing all such automorphisms, which seems to be due to the author of the present review: see *Bulletin of the American Mathematical Society*, volume 6 (1900), page 337. The statement that when the multiplying group is the same as the group itself each element of the group corresponds to its square is clearly not generally true as may be seen by considering such automorphisms of the abelian group of order 2^m and of type $(1, 1, 1, \cdots)$.

Chapters 10 to 14 are devoted mainly to linear substitution groups and occupy more than one third of the space of the book, while chapter 15 is devoted to the theory of equations. The final chapter begins with a brief account of the Lagrange theory of equations. This is followed by the Galois theory and by applications of general group theory and substitution groups. In view of the

advanced character of the book, readers may be somewhat surprised to find so much emphasis on the literature in the German language. For instance, in the preface it is stated that the readers who found the presentation too brief might consult the books by Weber and Netto for more complete treatments.

On the whole, the volume under review furnishes a very attractive introduction into some of the most modern developments of the theory of groups of finite order, with emphasis on its applications, and we can only wish it success along with the other volumes of the interesting series to which it belongs now being published under the general editorship of R. Courant of the University of Göttingen.

G. A. MILLER.

A Treatise on the Theory of Bessel Functions. By G. N. WATSON. Cambridge University Press, 1922. ii + 804 pages. Price $22.50.

"This book has been designed with two objects in view. The first is the development of applications of the fundamental processes of the theory of functions of complex variables. For this purpose Bessel functions are admirably adapted; while they offer at the same time a rather wider scope for the application of parts of the theory of functions of a real variable than is provided by trigonometrical functions in the theory of Fourier series.

"The second object is the compilation of a collection of results which would be of value to the increasing number of mathematicians and physicists who encounter Bessel functions in the course of their researches. The existence of such a collection seems to be demanded by the greater abstruseness of properties of Bessel functions (especially of functions of large order) which have been required in recent years in various problems of mathematical physics."

Bessel functions are perhaps from the point of view of applied mathematics the most important transcendants after the exponential group and their inverses.

It is this importance that justifies this great treatise on their theoretic properties and while the applied mathematician might have wished for illustrations of their use in physical problems, such additions would have vastly increased the size of the book and proved disappointing to those largely interested in the theoretic aspects of these functions. That Professor Watson has examined a large amount of material is shown by the index of titles of more than seven hundred papers and an author index of more than three hundred names of which among the sixteen American mathematicians the late Professor Bôcher's stands out preëminent. The book does ample justice to the large contributions of Sturm, Lommel, Carl Neumann, Sonine and Kapteyn who have created so much of this theory.

It is unfortunate that so many different Bessel functions of the second kind have been used by different mathematicians and still more unfortunate that the often elegant identities involving these functions cannot be expressed in terms of one type without loss of simplicity.

It suffices to say that this book is a rich mine of methods and results and is full of interesting problems of which we might mention the transcendence of the non-zero roots of $J_n(x) = 0$ when n is rational.

Professor Watson deserves the gratitude of his colleagues for the able and careful manner in which he has brought together this large amount of material

as well as for the well-balanced presentation of a subject which owes much to his own personal investigations.

It is unnecessary to say that the book work of the Cambridge Press is beyond all praise and is absolutely unrivalled elsewhere, fine paper, beautiful printing and wide margins only marred by a somewhat flimsy binding.

The book should certainly find a place, despite the high price, in every good mathematical library, but the cost will doubtless prevent its purchase by many persons desiring to own a copy of it.

M. B. PORTER.

Algebraic Numbers. Report of the Committee on Algebraic Numbers (National Research Council), by L. E. DICKSON, H. H. MITCHELL, H. S. VANDIVER, and G. E. WAHLIN. The National Academy of Sciences, Washington, D. C., February, 1923. Paper, 8vo. 96 pages. Price $1.50.

The first object of this paper is to bring up to date the extensive report on the theory of algebraic number fields by D. Hilbert, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 4, 1894–5, pp. 175–546; French translation in *Annales de la Faculté des Sciences de Toulouse*, series 3, vol. I, 1909, 257–328; II, 1910, 226–456; III, 1911, 1–62 (notes by Humbert and Got, and errata). This supplementary report deals with the articles subsequent to Hilbert's report and such earlier papers as were not cited by him.

The second object of the paper is to deal with the literature, not in Hilbert's report, on fields of functions and related topics, such as Hensel's p-adic numbers and modular systems. Some of these topics have already been treated in the French *Encyclopédie*. In two special cases mentioned explicitly the present report leaves it to the reader to get the requisite information from the French article; otherwise, it is entirely independent of that article. Hence, with these two exceptions, the present report (taken with that of Hilbert) is intended to exhaust the literature of the subjects named.

Great care has been taken to make the list of references wholly complete; and a most painstaking search of the literature has been made with this end in view. In this respect this paper and the report of Hilbert give a full direction to the literature of that part of the theory of numbers which is not being covered in Dickson's monumental *History of the Theory of Numbers*. The importance of this report is greatly enhanced by its having this relation to that comprehensive work.

In the treatment of the forty-five topics which form the subject matter of as many articles into which the four chapters are divided there is a great variety of procedure ranging from a mere listing of references to articles at one extreme to an outline of the theory at the other extreme. In the chapter on Hensel's p-adic numbers there is a careful outline of the leading ideas with a survey of the literature and a statement of the more important theorems. In such topics as cubic fields, Galois fields, Abelian fields, and units of a general field there is but little more than a list of references.

From the supplementary character of the report it is inevitable that it should be rather fragmentary in character in many places; for it contains merely the necessary additions to a previous well-made report of great importance. But this does not decrease the value of the paper to a man of research who wishes to get access to the whole literature of the subject treated. The devotees of the theory of numbers owe a distinct debt of gratitude to the authors of this report for their difficult and careful labor in bringing together and systematizing the literature of their subject not already cited in the report of Hilbert.

R. D. CARMICHAEL.

Frequency Arrays. By H. E. SOPER. Cambridge University Press, 1922. 48 pages.

"Since in other branches of science symbols bearing an objective or logical significance have been usefully employed, conjoined with symbols of number, to express quantity, it may be expected that in the science of statistics, which of its essence is the enumeration of logical classes, such symbols will find serviceable application.

The aim, then, has been to recommend the use of logical symbols in the enumeration of logical classes. It has been no material part of the purpose to establish new formulæ and results in the mathematical theory of statistics and if new conclusions have been reached these will serve chiefly to help point the precept, since no difficult analysis has been undertaken or is anywhere involved.

It is possible that, with wider familiarity in their use, the applications of the symbols of denomination may bear extension and that they may be found of assistance in the development of the higher theory both of statistical and other distributions."

United States Life Tables—1890, 1901, 1910, and 1901–1910. By JAMES W. GLOVER, expert special agent of the Bureau of the Census. Washington, Government Printing Office, 1921. 4vo. 496 pages. Price $1.25.

Mr. Sam L. Rogers, director of the Census, writes in the "Letter of Transmittal": "The Bureau has had the advice and coöperation of a special census committee representing the Actuarial Society of America, composed of John K. Gore, chairman, Robert Henderson, Arthur Hunter, William A. Hutcheson, and Henry Moir. The tables have been prepared along lines approved by this committee."

It is difficult to describe in a few words a book that contains so much valuable information, a book valuable not only to actuaries and statisticians but to the general public—which must have some interest in vital statistics, in life insurance, in old-age pensions, in the settlement of estates involving life interests, and in the settlement of claims for injury or death. Indeed, Part I, consisting of 25 large pages, is a "non-technical description and explanation of life table functions, graphs, and other parts of text and tables": it gives in a particularly lucid manner, by the question-answer method, with continual reference to specific tables, just the information sought by the general public. For the United States Tables (Part II), the classifications, in addition to dates, are: Male and female, white and negro, native and foreign-born white, urban and rural. There are tables for the "selected registration states," Indiana, Massachusetts, Michigan, New Jersey, and New York; and for the cities of Boston, Chicago, New York, and

Philadelphia. Part III compares in tabular form life functions in the United States Tables with those for Australia, Denmark, England, France, Germany, Holland, India, Italy, Japan, Norway, Sweden, and Switzerland. Comparisons are also made with the rates found in ten important tables based upon the experience of insurance companies in America, Europe, and Japan. The graphs of life table functions, given in Part IV, are especially informing. Part V gives the usual tables for life insurance premiums, annuities, and the commutation columns used in practical problems involving money—with interest rates from 3 per cent. to 6 per cent.

For the teacher of mathematics, the concluding three parts will be of most interest. Part VI gives the requisite mathematical theory. Part VII describes in minute detail the construction of the life table for Males in the State of New York, 1910, with photographs of adding machine tapes—this is followed by an outline of the process for each life table. Part VIII gives the statistics used as data for each table. These three parts form a valuable laboratory manual for drill in substantial statistical work of college grade. With the use of calculus, Part VI describes the relations between the two aggregates of the living and the three aggregates of the dead, and deduces the formulas required for the computation. For graduation, the osculatory fifth-difference formula is derived; and annual differences are expressed in terms of quinquennial differences. Almost all of this can be presented in substance to a class of some maturity without presupposing calculus. And, using the New York Table computations as a model, the class can construct one of the other tables. In Tape 158, p. 381, it seems desirable—for class work—to have a check before age 99 is reached. For this purpose, note that 99,906,624, fourth number above last in Col. 1, p. 379, diminished by three times the number above it, and increased by 90,208, first number in Col. 2, yields the number marked ⑤ in Col. 3, p. 381. This same form of check can be applied to the subtotals in Tape 153. The horizontal scheme of work on page 382 was made over into a vertical one by the reviewer for class use. The construction of the tapes is described with such care that a student of moderate ability should find no difficulty in repeating the steps. The presentation of the mathematical theory is likewise excellent. The reviewer was unable to see the utility of Equation (31), p. 344, inasmuch as (38) is not a form of (31), as seen by setting $x = -1$: but this is a minor matter.

The book concludes with an index of 20 pages,—some idea of the thoroughness of this index can be gained from the fact that about 3,000 lines are used in columns $3\frac{1}{2}$ inches wide. The printing of the book is pleasing to the eye; and the headings of the tables are obviously the result of great study and painstaking—the headings specifying in minute detail the exact contents of the columns. A teacher of statistics will find this book very hard to match as a thoroughgoing exposition of the details of life-table construction. From the Government Printing Office, the book can be obtained for $1.25,—a price which hardly seems to pay for the paper used.

EDWARD L. DODD.

NOTES ON RECENT PUBLICATIONS.

In Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften zu Leipzig, mathematisch-physische Klasse, 1922, volume 44, no. 3, pages 157–160, there is a "Nachruf" by G. Herglotz, "Zum Gedächtnis Johannes Thomae 1840–1921" (*1921*, 336).

A French translation of WOODS and BAILEY'S *Analytic Geometry and Calculus* (Boston, Ginn, 1917) is now in the press and permission has been asked to translate *A Course in Mathematics for Students of Engineering and Applied Science* (2 volumes, Ginn, 1907–1909) by the same authors.

Under the auspices of the Society of Sciences in Göttingen, volumes I–X_1 of Gauss' Werke were published 1863–1917. The second part of volume X (Aufsätze über Gauss' wissenschaftliche Tätigkeit auf den Gebieten der reinen Mathematik), the first part of volume XI (Nachlese und Briefwechsel zur Physik, Astronomie und Chronologie) and the second part of volume XI (Aufsätze über Gauss' wissenschaftliche Tätigkeit auf den Gebieten der reinen Mathematik) are in the press. Volumes X_2 and XI_2 are to be composed of single monographs separately paged. We have already referred (*1921*, 79) to the first edition of eight of these as they appeared in the Göttinger Nachrichten, 1911–1920. Two parts of volume X_2 have been published. The first (1922) contained Bachmann's "Ueber Gauss' zahlentheoretische Arbeiten" (1911) and Oskar Bolza's "Gauss und die Variationsrechnung" (95 pages not previously printed). The second part (1923) contains (123 pages) an almost unchanged reprint of Stäckel's essay on "Gauss als Geometer" (1917) which the late author had intended should be much extended. The next essay published is to be Schlesinger's "Ueber Gauss' Arbeiten zur Funktionentheorie" (1912) completely rewritten. Volume XI_2 is to contain the essays in Astronomy by Brendel (1919), on Physics by Schäfer, on Geodesy by Galle and on "Gauss als Zahlenrechner" by Männchen (1918).

The first number of *The Quarterly Journal of Pure and Applied Mathematics*, volume 49, appeared in October, 1920; the fourth number issued in March, 1923, contained a paper "Singly infinite class number relations" (pages 322–327) by E. T. BELL, of the University of Washington. We have already referred (*1921*, 137; *1922*, 224) to Professor Bell's other papers in the volume. Number 3 (October, 1922) contains "Abstract definitions of the symmetric and alternating groups and certain other permutation groups" (pages 226–283) by R. D. CARMICHAEL, of the University of Illinois.

We have already referred (*1921*, 317–318; *1922*, 222) to *Fundamenta Mathematicæ*, the remarkable annual periodical dealing wholly with the theory of aggregates, and published in Warsaw, Poland. Volume 4 (4 + 372 pages) appeared in April, 1923, and contains two papers by Americans: "On the generation of a simple surface by means of a set of equicontinuous curves" by R. L. MOORE, 106–117; "Note on a paper of M. Banach" by N. WEINER, 136–143. The extremely low subscription price (20 French francs, less than $1.40) is in happy contrast to the extortionate sums demanded by the Germans

for their periodicals. American libraries should encourage this notable publication by their subscriptions.

On the two hundredth anniversary of the birth of Euler, a committee of the *Society of Swiss Naturalists* launched the project of international coöperation for the publication of his collected works. Academies and individuals subscribed for about 300 sets, and about one hundred thousand (100,000) Swiss francs were collected, for the most part in Switzerland; the American Mathematical Society subscribed five thousand (5,000) francs. Eighteen (18) of the estimated seventy (70) volumes have been published.

By reason of the European War, nearly one half of the subscribers have been unable to meet their obligations in full. Under these circumstances, a considerable number of new subscribers must be secured if the completion of the undertaking is to be possible in the near future. There are three plans for subscribing. Those libraries or individuals wishing for information with a view to joining in promoting this great international undertaking should communicate with the Official Representative of the Euler Committee for the United States and Canada, Professor R. C. ARCHIBALD, Brown University, Providence, R. I.

ARTICLES IN CURRENT PERIODICALS.

ANNALES DE L'ÉCOLE NORMALE SUPÉRIEURE, volume 58, January, 1923: "Les modules des zéros des polynomes" by M. P. Montel, 1–32.

ANNALI DI MATEMATICA, series 3, volume 31, March, 1922: "Nuove ricerche sulle corrispondenze algebraiche fra i punti di una curva algebraica" by C. Rosati, 1–49; "Determinazione delle ipersuperficie che ammettono rappresentazioni geodetiche" by E. Bompiani, 51–80; "Sulle equazioni integrali non lineari" by A. Vergerio, 81–119; "Sopra due teoremi di Dirichlet" by A. M. Bedarida, 121–125; "Intorno alle involuzioni situate sopra le superficie iperellittiche con due fasci di curve ellitiche" by N. Spampinato, 127–148; "I gobbo-circolanti e i divisori di zero di un particolare sistema di numeri complessi ad n unità" by V. Amato, 149–164. November, 1922: "Riavvicinamento di geometrie differenziali delle superficie: metriche, affine, proiettiva" by G. Sannia, 165–189; "L'intorno d'un punto d'una superficie considerato dal punto di vista projettivo" by E. Cech, 191–206; "Sulle serie di polinomi di una variabile complessa. Le serie di Darboux" by N. Abramescu, 207–249; "I fondamenti della geometria proiettivo- differenziale secondo il metodo di Fubini" by E. Cech, 252–278; "Sull' indipendenza di un integrale da-parametri nel caso più generale" by G. Usai, 279–294.

BULLETIN DES SCIENCES MATHÉMATIQUES, series 2, volume 47, February, 1923: "Réduction des systèmes algébriques de points appartenant à une même courbe algébrique. Théorème d'Abel" by B. Gambier, 76–96.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 29, January, 1923: "The October meeting of the Society" by R. G. D. Richardson, 1–10; "The October meeting of the San Francisco Section" by B. A. Bernstein, 10–13; "The Frank Nelson Cole prize in algebra" by R. G. D. Richardson, 14; "Periodic solutions in the problem of three bodies" by F. H. Murray, 15–16; "Note on quartiles and allied measures" by D. Jackson, 17–20; "Ruled surfaces with director planes" by J. K. Whittemore, 21–25; "On transformable systems and covariants of algebraic forms" by C. C. Macduffee, 26–33; Reviews: by A. J. Kempner of O. Perron, *Irrationalzahlen* (Berlin and Leipzig, 1921), 34–36; and of P. Bachmann, *Grundlehren der neueren Zahlentheorie* (Berlin and Leipzig, 1921), 36–37; by A. B. Coble of H. Malet, *Étude Géométrique des Transformations Birationelles et des Courbes Planes* (Paris, 1921), 38; by J. W. Young of E. H. Neville, *The Fourth Dimension* (Cambridge, 1921), 38; and of G. Kowalewski, *Mathematica Delectans* (Leipzig, 1921), 40; by P. J. Daniell of L. Page, *An Introduction to Electrodynamics* (Boston, 1922), 39, and of R. Gramel, *Der Kreisel* (Braunschweig, 1920), 40; and by H. L. Rietz of E. Czuber, *Wahrscheinlichkeitsrechnung* (Vol. II, 3d ed., Leipzig and Berlin, 1921) and *Die statistischen Forschungsmethoden* (Wien, 1921), 39; Notes, 41–42; New publications, 43–48—

February: "An uncountable, closed and non-dense point set each of whose complementary intervals abuts on another one at each of its ends" by R. L. Moore, 49–50; "Total geodesic curvature and geodesic torsion" by J. K. Whittemore, 51–54; "The name 'divergent' series" by F. Cajori, 55; "A qualitative definition of the trigonometric and hyperbolic functions" by P. Franklin, 56–64; "Groups in which the number of operators in a set of conjugates is equal to the order of the commutator subgroup" by G. A. Miller, 64–70; "On curves kinematically related to a given curve" by H. Poritzky, 71–78; Reviews: by D. E. Smith of T. L. Heath, *A History of Greek Mathematics* (2 vols., Oxford, 1921), 79–84; by L. W. Dowling of F. Amodeo, *Lezioni di Geometria Proiettiva* (3d ed., 2d reprint, Naples, 1920), 85–86; by H. L. Rietz of J. W. Glover, *United States Life Tables 1890, 1901, 1910 and 1901–10* (Washington, 1921), 86; and of F. Insolera, *Lezioni di Statistica Metodologica* (Turin, 1921), 90; by K. W. Lamson of A. S. Eddington, *Espace, Temps et Gravitation*, French translation by J. Rossignol (Paris, 1921), 87; by W. C. Graustein of W. de Tannenberg, *Conférences sur les Transformations en Géométrie Plane* (Paris, 1921), 87; by J. B. Shaw of G. Juvet, *Introduction au Calcul Tensoriel et au Calcul Différentiel Absolu* (Paris, 1922), 88–89; by E. W. Brown of *Annuaire du Bureau des Longitudes pour l'An 1922* (Paris, 1922), 89; by A. Emch of M. Groll, *Kartenkunde* (Berlin and Leipzig, 1922), 89–90; and by E. B. Lytle of B. Branford, *A Study of Mathematical Education* (2d ed., Oxford, 1921), 90; Notes, 91–92; New publications, 93–96.

Journal de Mathématiques Pures et Appliquées, series 9, volume 2, no. 1, 1923: "Differential properties of functions of a complex variable which are invariant under linear transformations" by E. J. Wilczynski, 1–51. The first part appeared in this journal, series 9, volume 1, part 4, 1922. "Sur un ensemble non mensurable B" by N. Lusin and W. Sierpinski, 53–72; "Recherche des systèmes cycliques de triples de Steiner différents pour N premier (ou puissance de nombre premier) de la forme $6n + 1$" by S. Bays, 73–98.

Mathematics Teacher, volume 15, October, 1922: "The strength of the mental connections formed in algebra" by E. L. Thorndike, 317–331; "Fundamental principles of algebra" by R. L. Modesitt, 332–346; "Rabbi Ben Ezra on permutations and combinations" by J. Ginsburg, 347–356; "Introducing Mechalus to geometry" by Mary A. Potter, 357–360; "Note on Mr. Evans's paper in the March Teacher" by H. F. Hart, 360–361; "Original solution in plane geometry" by R. A. Laird, 361–364; "Professor Hedrick's report on the function concept in elementary mathematics" by H. E. Webb, 364–368; "News and notes," 369–374; "Research department," 375–376; "New books," 377–379—November: "The case for general mathematics" by W. D. Reeve, 381–391; "Errors in computations and the rounded number" by H. Rice, 392–404; "The constitution of algebraic abilities" by E. L. Thorndike, 405–415; "Romance in science" by Bessie I. Miller, 416–422; "Some mathematics of the calculating machine" by L. L. Locke, 423–428; "Mathematical clubs in the high school" by Sophia Refior, 434–435; "Recent articles of interest to mathematics teachers" by N. R. Howell, 435–439; "News and notes," 440–443—December: "Non-Euclidean geometry" by W. H. Bussey, 445–459; "The study of mathematics under the individual system" by Mary M. Reese, 460–466; "Problems concerning the teaching of secondary mathematics" by A. Davis, 467–477; "The future development of mathematical education" by C. N. Moore, 478–483; [Quotation: "In addition to bringing home to the student the wide use of mathematical knowledge in the activities of the modern world, we must also give him some notion of its origin and growth and its important rôle in the development of our civilization. We must not let him rest under the impression that mathematics was invented in order to provide intricate and vexatious puzzles for the adolescent mind. We must demonstrate to him that man was led to the pursuit of mathematical knowledge by his eager desire to understand the universe and to control the forces of nature, that he found the knowledge essential for the higher developments of trade and commerce and all the other varied developments that have had a place in the creation of our present-day civilization, in short that the progress of the world is now and always has been bound up with the development of our knowledge of mathematics."] "The function concept in high school mathematics" by J. M. Kinney, 484–495; "An historic theorem in plane geometry" by W. H. Carnahan, 496; "Geometry speaks" by Eva M. Palmer, 496–500; Review by A. S. Otis of The Thurstone Vocational Guidance Tests: Arithmetic, Algebra, Geometry (World Book Company, Yonkers, N. Y.), 506–507.

Messenger of Mathematics, volume 52, no. 4, August, 1922: "Notes on some points in the integral calculus" by G. H. Hardy, 49–53; "The dissection of rectilinear figures" by W. H. Macaulay, 53–56; "Relations between the numbers of Bernoulli, Euler, Genocchi, and Lucas" by E. T. Bell, 56–64.

Nouvelles Annales de Mathématiques, volume 80, October, 1922: "Sur l'étude

algébrique des problèmes de division des arcs" by E. Vessiot, 1–4; "Sur la trisection d'un angle" by B. Niewenglowski, 4–8; "Sur une notion d'equivalence locale apte à préciser certains points de la théorie des enveloppes" by G. Bouligand, 8–21; "Sur les ombilics" by P. Montel, 21–23—November, 1922: "Sur une manière simple d'obtenir géométriquement les formules de Lorentz" by M. Morand, 41–49; "Introduction à l'étude de la mécanique et de ses principes" by G. Bouligand, 50–58; "Sur la conservation de la courbure geodésique dans la déformation d'une surface" by R. Brichard, 58–61; "Sur les coniques focales" by C. Bioche, 62–63; "Remarques sur les trièdres" by C. Bioche, 63–65—December, 1922: "Monographie des polynomes de Kummer" by P. Humbert, 81–92; "Introduction à l'étude de la Mécanique et de ses principes" (continued) by G. Bouligand, 93–109; "Notes sur les podaires" by M. F. Egan, 109–112—January, 1923: "Système harmonique de trois coniques" by J. Lemaire, 121–135; "Introduction à l'étude de la mécanique et de ses principes" (continued) by G. Bouligand, 135–147; "Démonstration de la formule de l'accélération dans le mouvement relatif" by B. Niewenglowski, 147–150; "Remarques sur les Jacobiens" by C. Bioche, 150–153; "Developpables formées avec les normales d'une quadrique" by L. de la Roëre, 153–159—February, 1923: "Un théorème sur les équations algébriques entières" by G. Casabonne, 161–165; "Remarques au sujet d'un des problèmes de mécanique donnés au concours d'agrégation en 1921" by R. Thiry, 165–172; "Sur deux familles de courbes orthogonales" by G. Fontené, 173–180; "Introduction à l'étude de la mécanique et de ses principes" (concluded) by G. Bouligand, 181–188.

SCIENCE, volume 56, December 22, 1922: "The Frank Nelson Cole prize in algebra" by R. G. D. Richardson, 710–711—December 29: Review by R. Pearl of *United States Life Tables 1890, 1901, 1910 and 1901–1910* prepared by J. W. Glover (Washington, Bureau of the Census, 1921), 756–757—Volume 57, January 26, 1923: Reports of the Boston meetings of Section A (Mathematics) of the American Association for the Advancement of Science (by W. H. Roever), of the American Mathematical Society (by R. G. D. Richardson) and of the Mathematical Association of America (by W. D. Cairns), 107–108—February 2: "Geometry and physics" by O. Veblen [address as retiring vice-president and chairman of Section A (Mathematics) of the American Association for the Advancement of Science, Boston, December, 1922], 129–139—March 2: "The American Mathematical Society" [report of the Chicago Section meeting at Evanston, December 29, 1922] by A. Dresden, 276—March 23: "The American Mathematical Society" [report of the meeting at New York, February 24, 1923] by R. G. D. Richardson, 364.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 23, April, 1922 [published April, 1923]: "Some generalizations of geodesics" by E. J. Wilczynski, 223–239; "On the gyroscope" by W. F. Osgood, 240–264; "The relative distribution of the real roots of a system of polynomials" by C. F. Gummer, 265–282; "A general theory of conjugate nets" by E. P. Lane, 283–297; "Parallel maps of surfaces" by W. C. Graustein, 298–332.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 53, nos. 11–12, published December 5, 1922: "Die zeichnerische Ermittlung der Konstanten eines Luftlichtbildes, sowie Herstellung des Grund- und Aufrisses von Geländepunkten bei unebenem Gelände" by M. Brettar, 249–257; "Über den Anfangsunterricht in der Differentialrechnung" by K. Bögel, 257–258; "Zur Gruppierung der Vierecke" by A. Gottschalk, 259–260; "Einige Anwendungen der Spiegelung" by W. Weber, 264–265; "Die Mathematikertagung in Leipzig" [report of the annual meeting of the Deutsche Mathematiker-Vereinigung, September 17–22, 1922] by M. Zacharias, 270–272; "Bücherbesprechungen," 274–279; "Zeitschriftenschau," 279–281—Volume 54, no. 1, published January 30, 1923: "Ergänzungen zur Elementarmathematik" by W. F. Meyer, 1–10; "Die Lehre von den Wurzeln im elementaren Unterricht" by K. Kommerell, 10–21; "Graphische Darstellung von Sätzen der elementaren Algebra" by E. Dintzel, 21–26; "Elementares Verfahren zur Bestimmung der Elektricitätsverteilung auf dem Ellipsoid und zur Ermittlung der Kapazität einer Scheibe und eines Stabes" by H. Dörrie, 26–29; "Physik und Mathematik auf der Oberstufe von Vollanstalten" by K. Hahn, 30–34; "Zur Einführung der komplexen Zahlen im Mittelschulunterricht" by S. Weich, 34–36; "Die artilleristische Aufgabe im Unterricht der analytischen Geometrie" by H. Semiller, 37; "Zur Empfindlichkeit der Wage" by Janss, 38–39; "Aufgabenrepertorium," 39–45; "Bücherbesprechungen," 53–61; "Zeitschriftenschau," 61–62.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

The Cornell Parabola, Cornell University, Ithaca, N. Y.

[*1920*, 479.]

For membership in the Cornell Parabola, the mathematics faculty and students who have taken elementary calculus are eligible. As officers for the year 1922–1923, the following were elected: President, John Wood '24; secretary-treasurer, Violet Holloway '23; member of the executive committee, Joe Nobile '25; faculty adviser, Professor H. M. Morse. The club met at 7:30 p.m., as follows:

November 17, 1922: "Theory of numbers" by Professor W. L. G. Williams.
December 7: "The Parabola" by Ida Itzkowitz, Gr.
December 16: Christmas party at the home of Professor and Mrs. F. B. Owens.
February 15, 1923: "The mathematics of navigation" by John Wood '24.
March 30: "Scales of notation" by H. I. Lane, Gr.
April 25: "Einstein's theory of relativity" by Professor Morse. An open meeting with an attendance of nearly 200.
May 24: "Modern mathematics in Italy" by Professor Virgil Snyder.

(Reported by Miss Holloway.)

The Euclidean Circle of the University of Indiana, Bloomington, Ind.

[*1918*, 228.]

As officers for 1922–1923, the following were elected: President, Baker Hindman '23; vice-president-treasurer, Joseph Sheedy '23; secretary, Trula Sidwell '24. Meetings were held as follows:

December 11, 1922: "Mathematical recreation" by Professor Cora B. Hennel.
December 15: "Life of Isaac Newton" by Edna Ellis '23; "The famous problems of antiquity" by George Murphy '23.
February 6, 1923: "A paper on π" by Josephine Rich '23.
February 19: "Unified mathematics" by Joseph Sheedy '23.
March 12, 1923: "Probability" by Marie Sangennebo '24 and Jessie McAtee '23.
March 26: "Mathematics of the calendar" by Avie Burkett '23; "Mathematics of astronomy" by Henry Yarbrough, instructor.
April 17: "Women and mathematics" by Agnes McLeaster '24.
April 30: "Concurrent lines and collinear points" by Lionel Martin '24; "Proof that the angle cannot be trisected" by Mary Winget '23.
May 14: "Systems of notation" by Earl Klinger '24; "Mathematical prodigies" by Dorothy Munns '24.

(Reported by Miss Sidwell.)

The Mathematics Club of Iowa State Teachers College, Cedar Falls, Ia.

[*1920*, 29.]

With an average attendance of about thirty-five members, meetings were held in 1922–1923, as follows:

October 18, 1922: "New solutions for equations of higher order" by Professor C. W. Wester.
November 8: "The Burroughs calculating machine and its inventor" by Professor C. A. Speer,—with a demonstration of the machine.
November 22: "Construction of a parabolic arch" by B. A. Jenson '24; "Velocity of the earth in its orbit" by Harrietta Shrimp '25; "The golden section and the use of conic sections" by Merrill Muzzey '23; "The area and the construction of an ellipse" by Mildred Chatterton '23; "Reflectors and whispering galleries" by Harold White '24.
January 17, 1923: "A four step solution of the general form of the quadratic equation" by Myrtle Frederickson '25; "Introduction of logarithms" by Bessie Hutchinson '25, Edward Bark '25, and Vale Dunn '26; "The W. A. G. plane" by Professor R. G. Dougherty.
February 7: "Radio-activity" by Professor Kadish.

February 28: "Mathematical drills and recreations" by Professor E. E. Watson.
April 4: "Correlated mathematics in Iowa" by Professor I. S. Condit.
May 9: "The content of a course in freshman mathematics" by Professor Watson.

(Reported by Professor Watson.)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF MAINE, ORONO, ME.
[*1918*, 453.]

The following officers were elected for the year 1922–1923: President, Helen Shorey '23; vice-president, Edwin Hadlock '24; secretary and treasurer, Vera Savage '24; faculty adviser, Mr. Frank S. Beale, instructor. The following papers were read:
October 4, 1922: "History of the Club" by Dean J. N. Hart; "Mathematics clubs" by Professor N. R. Bryan.
October 25: "Rithmomachia" by Chester Austin '23.
November 8: "Reorganization of mathematics in secondary schools" by Professor Bryan.
November 22: "Another world" by Edwin Hadlock '24.
December 13: "The use of the sine function expansion in determining the area of an egg-shaped figure" by Mr. Warren Loving, instructor.
January 10, 1923: "A class of seven elements as applied to algebra" by Helen Shorey '23, retiring president. Chester Austin '23 was elected president for the second semester.
February 14: "Recreations in mathematics." General discussion.
February 28: "The vector method used in proving some propositions in geometry" by Mr. Warren Lucas, instructor.
March 14: Repetition of the talk given by Mr. Lucas at the last meeting.
April 12: "The fourth dimension and the Bible" by Ethelyn Percival '24.
April 26: "Interpolation" by Professor H. R. Willard. The following were elected as officers for the coming year: President, Vera Savage '24; vice-president, Ethelyn Percival '24; secretary and treasurer, Donald Trouant '25; faculty adviser, Professor Bryan.
May 15: A short play on the fourth dimension was presented by Misses Harkness, Percival, Savage, and Shorey. Professor Bryan reviewed the work of the club for the current year, and gave suggestions for the coming year.

(Reported by Miss Savage.)

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OREGON, EUGENE, ORE.
[*1918*, 134.]

As officers for the year 1922–1923, the following were elected: President, R. M. Elliot, Gr.; vice-president, Virl Bennehoff '23; secretary, Wave Lesley '23; treasurer, Ted McAlister '23; members of the executive committee, Gertrude Tolle '23, Don Wilkinson '23, and Willa Loomis '24. Meetings were held as follows:
March 15, 1922: "The theory of numbers" by Professor W. E. Milne.
April 19: "Flat Land" by Laura Hammer, Gr.; "The fourth dimension" by R. M. Elliot, Gr. Election of officers.
May 18: "Nomograms" by Professor McAlister, of the Department of Mechanics and Astronomy. Problem solving.
June: Picnic.
October 31: "Japanese mathematics" by Professor E. E. DeCou.
November 15: Mystery program of puzzles, stunts, and charades by Gertrude Tolle '23 and Wave Lesley '23.
December 8: "Einstein's theory of relativity" by Professor L. L. Smail, of the University of Washington.
January 31, 1923: Models for physics and mathematics displayed and explained by Professor W. P. Boynton, of the Department of Physics.
February 22: Social evening at the home of Professor and Mrs. DeCou.
April 18: "Gamma functions" by Ted McAlister '23.
May 8: "Concrete bridges" by Professor McAlister. Election of officers for 1923–1924: President, Wave Lesley '23; vice-president, Sylvia Veatch '25; secretary, Mary Search '25; treasurer, Vera Hughes '25.
May 29: Picnic and installation of officers.

(Reported by Miss Lesley.)

THE TULANE MATHEMATICAL SOCIETY OF TULANE UNIVERSITY, NEW ORLEANS, LA.

The membership of this society includes many of the mathematics teachers of New Orleans. As officers for 1922–1923, the following were elected: President, Professor H. E. Buchanan; secretary-treasurer, Professor Anna M. Howe, of Newcomb College. Papers were presented as follows:

Fall meeting: "Applications of the binomial theorem" by Professor Howe.

Winter meeting: "Trade mathematics" by Mr. J. P. McGuire, teacher in mathematics at Delgado Trade School, New Orleans.

Spring meeting: "The motion of a heavy particle on the surface of a torus" by Mr. F. G. Eberle, teacher in mathematics at the Warren Easton Boys High School, New Orleans.

The Newcomb Science Club of Newcomb College, New Orleans, La., was organized in the fall of 1922 to stimulate interest in science and to correlate the work of the departments of science. As officers were elected: President, Odessa Lastrapes '23; vice-president, Beatrice Cosgrove '23; secretary-treasurer, Vivia de Milt '23. Papers on general science are read at the monthly meetings. At the December meeting, Carlotta Kraft '24 read a paper: "A report of the Einstein expedition to Christmas Island."

(Reported by Miss Lastrapes.)

THE WHITE MATHEMATICS CLUB, UNIVERSITY OF KENTUCKY, LEXINGTON, KY.
[*1922*, 417.]

October 4, 1922: Election of officers: President, Professor H. H. Downing; secretary, Professor Elizabeth Le Stourgeon.

October 19: "Some ancient methods of multiplication" by Dean P. P. Boyd; "Equivalence of equations" by Professor J. M. Davis.

November 2: "Fourier series" by Professor Le Stourgeon.

November 16: "Descartes' rule of signs" by Minnie Peterson '24; "Sturm's theorem" by H. Mobley '24.

November 29: "Vector treatment of the motion of a rigid body about a point" by Professor E. L. Rees.

January 10, 1923: "Force at a point due to a circular charge of electricity, the point being in the plane of the charge" by Professor M. N. States, of the Department of Physics.

February 14: "Solution of cubic equations" by Mary Gordon '25; "Solution of quartic equations" by Marion Brown '24.

February 28: "Foci" by Dean Boyd.

March 15: "Interference methods applied to astronomy" by Norman Beese '23.

April 5: "The analytical polygon" by Mr. R. V. Blair, instructor.

April 19: "The cissoid of Diocles" by Tomie Bronsbon '23; "The conchoid of Sluse" by W. Hutcherson, Gr.

May 3: "Sylvester's method of elimination" by Mr. George Seubert, instructor.

(Reported by Professor Le Stourgeon.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3035. Proposed by R. M. MATHEWS, Wesleyan University.

Generalize projectively and prove that the envelope of the bisectors of the angles between corresponding lines of two perspective pencils is a curve of the first class.

3036. Proposed by F. M. GARNETT, Savannah, Georgia.

The inside dimensions of a chest are $l \times w \times d$; find the greatest length of a rectangular piece of timber with the cross-section $a \times b$ which fits in the closed chest. Numerical application: $l = 6$ feet. $w = 3$ feet, $d = 2$ feet, $a = b = 2$ inches.

3037. Proposed by E. O. BROWER, Chicago, Illinois.

The angles A, B, C of a triangle are given. A logarithmic spiral is tangent to AB at B and to AC at C; at what angle does the curve cut each radius vector?

3038. Proposed by B. F. FINKEL, Drury College.

A fly starts from a point on the periphery of a right elliptic cone, altitude h and equation of the periphery of the base $b^2x^2 + a^2y^2 = a^2b^2$, and walks obliquely along the surface of the cone, crossing the elements of the cone at a constant angle, α. How far has the fly travelled when coming, for the first time, to the element of the cone passing through the point of departure?

3039. Proposed by J. K. WHITTEMORE, Yale University.

Given a conic S, a point A, and a line l. Through A is drawn a variable line cutting S in P and Q. Find the envelope of a conic which is tangent to S at P and Q and which is tangent to l.

SOLUTIONS.

2877 [1921, 89]. Proposed by J. B. REYNOLDS, Lehigh University.

A particle slides down the rough arc of a cardioid, $r = a(1 - \cos\theta)$, which lies in a vertical plane, the initial line being horizontal. If the coefficient of friction, μ, equals 1/3, find θ for the point where the particle leaves the curve, if it starts at $\theta = 90°$.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio, AND OTTO DUNKEL, Washington University.

Let v be the velocity of the particle at any point of the curve, φ the angle which the tangent at this point makes with the initial line, ρ the radius of curvature, R the reaction of the curve, and mg the weight of the particle. Resolving the forces along the tangent and the normal, we obtain for the equations of motion

$$mg\sin\varphi - \mu R = m\frac{dv}{dt}, \qquad mg\cos\varphi + R = -\frac{mv^2}{\rho}; \tag{1}$$

and, eliminating R from these two equations, we have

$$g(\sin\varphi + \mu\cos\varphi) = \frac{dv}{dt} - \mu\frac{v^2}{\rho}. \tag{2}$$

If s is the length of the curve measured from the cusp ($\varphi = 0$), then

$$-v = \frac{ds}{dt} = \frac{ds}{d\varphi}\frac{d\varphi}{dt} = \rho\frac{d\varphi}{dt}, \qquad \frac{dv}{dt} = \frac{dv}{d\varphi}\frac{d\varphi}{dt} = -\frac{v}{\rho}\frac{dv}{d\varphi}. \tag{3}$$

By use of (3) the equation (2) becomes

$$\frac{d(v^2)}{d\varphi} + 2\mu v^2 = -2g\rho(\sin\varphi + \mu\cos\varphi). \tag{4}$$

Now for the cardioid $\varphi = 3\theta/2$ and $\rho = (4a/3)\sin(\varphi/3)$, and after this value of ρ is inserted equation (4) may be written

$$d(v^2e^{2\mu\varphi}) = -\frac{8ag}{3}e^{2\mu\varphi}\sin\frac{\varphi}{3}(\sin\varphi + \mu\cos\varphi)d\varphi. \tag{5}$$

The integration of this equation gives

$$v^2 = ke^{-2\mu\varphi} + 2ag\left[\frac{5\mu\cos\frac{4\varphi}{3} + (2 - 3\mu^2)\sin\frac{4\varphi}{3}}{4 + 9\mu^2} + \frac{(3\mu^2 - 1)\sin\frac{2\varphi}{3} - 4\mu\cos\frac{2\varphi}{3}}{1 + 9\mu^2}\right]. \tag{6}$$

Inserting the value of φ above and $\mu = 1/3$, this equation becomes

$$v^2 = \frac{2ag}{3}[\cos 2\theta + \sin 2\theta - \sin\theta - 2\cos\theta + 2e^{(\pi/2)-\theta}], \tag{7}$$

where the constant k has been determined so that $v = 0$ when $\theta = \pi/2$. The particle leaves the curve at the moment when $R = 0$, and this value of R and v^2 above inserted in the second equation of (1) gives

$$\cos 2\theta + 2\sin 2\theta - 2\sin\theta - 2\cos\theta + 2e^{(\pi/2)-\theta} = 0. \tag{8}$$

If we set $f(\theta) = e^{\theta-(\pi/2)}(\cos 2\theta + 2\sin 2\theta - 2\sin\theta - 2\cos\theta) + 2$, then $f'(\theta) = e^{\theta-(\pi/2)}(5\cos 2\theta - 4\cos\theta)$. When $\theta = 0$, both $f(\theta)$ and $f'(\theta)$ are positive, and when $\theta = \pi/2$, both are negative. Now $f'(\theta)$ vanishes only once in this interval, *i.e.*, when $\cos\theta = (2 + \sqrt{54})/10$. Hence $f(\theta)$ vanishes once and only once in this same interval. This root is found to be $\theta = 1.36019$ or, in degrees, $\theta = 77° 56'$.

Also solved by F. L. WILMER.

2904 [1921, 277]. Proposed by N. P. PANDYA, Amreli, Kathiawad, India.

Pairs of tangents to a conic intersect on a fixed straight line; find the locus of the middle points of the chords of contact.

SOLUTION BY OTTO DUNKEL, Washington University.

Let P be the pole of the fixed line and suppose that the conic has a center C and a diameter $A'CA$ passing through P. Let M be the middle point of a chord through P. Draw the chord AN parallel to PM. Then CM produced bisects AN in M'. It follows then from similar triangles that

$$PM = \frac{PC}{AC}\cdot AM' = \frac{PC}{AC}\cdot\frac{AN}{2} = \frac{PC}{AA'}\cdot AN.$$

This shows that, as N describes the given conic, M describes a similar conic with the ratio of similitude PC/AA'.

In the case of a parabola $PC/AA' = \frac{1}{2}$ and the locus of M is also a parabola.

Also solved by T. M. BLAKSLEE, A. M. HARDING, WILLIAM HOOVER and A. PELLETIER.

2914 [1921, 326]. Proposed by HARRIS HANCOCK, University of Cincinnati.

If $a_1, a_2, \cdots, a_n$ are n positive integers, a_{ij} the greatest common divisor of a_i and a_j, d_m the greatest common divisor of all products of every m of these numbers ($m = 1, 2, \cdots, n-1$), then is

$$\prod_{ij} a_{ij} = d_1 d_2 \cdots d_{n-1}; \qquad (j > i;\ i = 1, 2, \cdots, n;\ j = 2, 3, \cdots, n).$$

In general, show that this theorem is true if $A_1, A_2, \cdots, A_n$ are any functions integral in any number of variables, with rational integral coefficients, or with algebraic coefficients.

Remark: This is a generalized statement in positive rational integers of the following theorem of much importance in the theory of algebraic numbers and due to Dedekind (Dirichlet, *Zahlentheorie*, Supplement XI): Let A, B, C be three moduls (Dedekind). Denote the greatest common divisor of A and B by $A + B$ and their product by $A \cdot B$. Dedekind proves that

$$(A + B)(B + C)(C + A) = (A + B + C)(AB + BC + CA).$$

Denote the greatest common divisor of two moduls A_i and A_j, that is, $A_i + A_j$, by A_{ij}; and write

$$D_1 = A_1 + A_2 + \cdots + A_n, \qquad D_2 = A_1A_2 + A_1A_3 + \cdots + A_{n-1}A_n,$$
$$D_3 = A_1A_2A_3 + A_1A_2A_4 + \cdots + A_{n-2}A_{n-1}A_n, \cdots,$$
$$D_{n-1} = A_2A_3\cdots A_n + A_1A_3\cdots A_n\cdots + A_1A_2\cdots A_{n-1}.$$

Show that

$$A_{12}A_{13}\cdots A_{n-1}A_n = D_1D_2\cdots D_{n-1}.$$

SOLUTION BY THE PROPOSER.

The following proof is found in one of my note books entitled "Miscellaneous." The proof is probably due to or suggested by *Frobenius*.

Let p be any prime integer that enters a_λ to the k_λ power and arrange the integers $a_1, a_2, \cdots, a_n$ so that

$$k_1 \leqq k_2 \leqq k_3 \leqq \cdots \leqq k_n.$$

Denote the greatest common divisor of a_i and a_j by a_{ij}. It is clear that p appears to the k_i power in a_{ij} and consequently to the power $(n-1)k_1 + (n-2)k_2 + \cdots + 1 \cdot k_{n-1}$ in the product Πa_{ij}, this product being defined in the statement of the problem. On the other hand p appears in d_1 to the k_1 power; in d_2 to the $(k_1 + k_2)$ power; $\cdots$ in d_{n-1} to the $(k_1 + k_2 + \cdots + k_{n-1})$ power.

With this the correctness of the formula as stated in the problem is established.

2916 [1921, 327]. Proposed by HARRIS HANCOCK, University of Cincinnati.

If p is any rational prime integer, and if $\alpha\ (\neq 1)$ is any root of $x^p - 1 = 0$, show that $p = P_1 \cdot P_2 \cdots P_{p-1}$, where $P_i\ (i = 1, 2, 3, \cdots p-1)$ are the ideals $(p, 1 - \alpha^i)$, which in turn may be reduced to principal ideals. [Remark: This is rather a good elementary example to show that an integer prime in one realm is factorable in a more extended realm.]

SOLUTION BY THE PROPOSER.

Let $\alpha_0 = 1, \alpha_1, \alpha_2, \cdots, \alpha_{p-1}$ be the roots of $x^p - 1 = 0$, so that

$$x^p - 1 = (x-1)(x-\alpha_1)\cdots(x-\alpha_{p-1}).$$

Differentiate with regard to x, and then put $x = 1$, with the result

$$p = (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{p-1}).$$

The proof of the problem is then immediate.

NOTE BY THE EDITORS: The final result is obtained by observing that, p being a prime integer > 2, $\alpha_i = \alpha^j$, where α is any root not equal to 1, $j \leqq p-1$, and no two j's are the same for different i's. Hence

$$p = (1-\alpha)(1-\alpha^2)(1-\alpha^3)\cdots(1-\alpha^{p-1}).$$

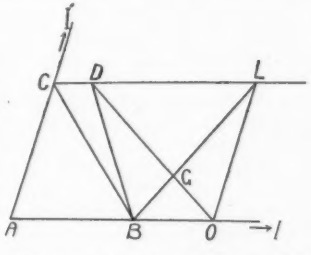
2921 [1921, 392]. Proposed by J. W. CLAWSON, Ursinus College, Pa.

ABC is a triangle cut by a transversal PQR, so that A, P; B, Q; and C, R are opposite vertices of a complete quadrilateral. Draw CD, PF, QE, chords in the circles circumscribing, respectively, triangles ABC, BRP, AQR, all these chords being parallel to AB.

Prove that (i) D, E, F are collinear; (ii) the line DEF passes through the Wallace point of the quadrilateral (the point of concurrency of the circles mentioned above); (iii) the line DEF intersects AB at the point of tangency to AB of the parabola which touches the four sides of the quadrilateral.

SOLUTION BY A. PELLETIER, Montreal, Canada.

Let O be the point of contact of AB with the parabola which touches the sides of the quadrilateral and let us consider the triangle ABC. Apply Brianchon's theorem to the hexagon $CB, BO, OI, IT, TI', I'C$ where I and I' are the points at infinity on AB and AC, respectively, and T is the point of tangency of the line at infinity. Then CI, OI' and BT meet in L and $ACLO$ is a parallelogram: also BL is parallel to the axis. Draw OG cutting BL in G so that $\angle GOB = \angle GBO$. We shall show that the point D where OG meets CL is the point D of the theorem. For $OD = BL$ and hence the triangles OBL and BOD are congruent: thus $\angle DBA = \angle CAB$ and the circle through A, B and C passes through D. From the properties of the parabola, W, the focus, must lie on OD since triangle GOB is isosceles. It is also known that the above circle passes through W. In the same manner it follows that E and F lie on OW.

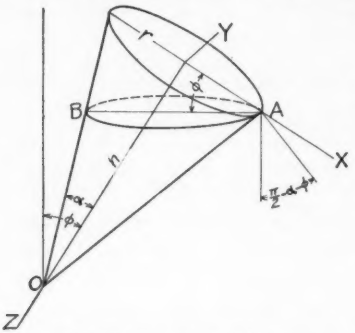


2936 [1921, 467]. Proposed by J. P. BALLANTINE, Chicago, Ill.

A person in drinking from a conical drinking glass tips it at a constant angular rate. At what angle will the delivery be the maximum and at what angle will the surface of the water be a maximum?

Solution by J. B. Reynolds, Lehigh University.

Let the height of the cone be h, its radius r and vertical angle 2α. Assume three mutually perpendicular axes, the x-axis in the plane of rotation of the axis of the cone, the y-axis perpendicular to this plane, the z-axis being the axis of the cone and the origin the center of the base.



We have then for the equation of the conical surface

$$r^2(h - z)^2 = h^2(x^2 + y^2), \tag{1}$$

and for the equation of the surface of the water when the axis of the cone makes an angle φ with the vertical

$$z = (r - x)\tan\varphi. \tag{2}$$

Now (1) and (2) give for the projection of the intersection of these two surfaces

$$r^2(h - r\tan\varphi + x\tan\varphi)^2 = h^2(x^2 + y^2),$$

which gives as a maximum width of this section

$$2y = 2r\sqrt{\frac{h - r\tan\varphi}{h + r\tan\varphi}} = 2r\sqrt{\frac{\cos(\varphi + \alpha)}{\cos(\varphi - \alpha)}}.$$

If AB is the length of this elliptic section, we have by the law of sines $AB\cos(\varphi - \alpha) = 2r\cos\alpha$, giving for the area

$$A = \tfrac{1}{2}\pi\overline{AB}\cdot y = \frac{\pi r^2\cos\alpha\cos^{1/2}(\varphi + \alpha)}{\cos^{3/2}(\varphi - \alpha)}$$

which is a maximum when $\tan(\varphi + \alpha) = 3\tan(\varphi - \alpha)$ or $\varphi = \tfrac{1}{2}\arcsin(2\sin 2\alpha)$. This indicates that the first surface is a maximum unless the vertical angle of the cone is less than 30° or greater than 150°. Now $\sin 2\varphi = 2\sin 2\alpha$; hence 2φ is between 2α and $180° - 2\alpha$ in value. Since physical considerations require that $0° < \varphi < 90° - \alpha$, it follows that $\alpha < \varphi$, and therefore $\alpha < 15°$.

Let the two values of φ obtained from $\sin 2\varphi = 2\sin 2\alpha$ be φ_1 and φ_2, φ_2 being the greater; then φ_1 makes A a minimum and φ_2 makes A a maximum. For $\alpha = 15°$ the values of φ_1 and φ_2 are each 45° so that for this limiting value we have A neither a maximum nor a minimum. As α decreases to 0°, φ_1 decreases to 0° and φ_2 increases to 90°. It remains to determine for what values of α the maximum given by φ_2 is greater than the first surface. For this condition we must have

$$\frac{\pi r^2\cos\alpha\cos^{1/2}(\varphi_2 + \alpha)}{\cos^{3/2}(\varphi_2 - \alpha)} > \pi r^2$$

which may be written $\cos^3\alpha\sin\alpha > \cos^3(\varphi_2 - \alpha)\sin(\varphi_2 - \alpha)$ since $\tan(\varphi_2 + \alpha) = 3\tan(\varphi_2 - \alpha)$. If now $x = \cos 2\alpha$, we have

$$\sin^2(\varphi_2 - \alpha) = \tfrac{1}{2}\{1 - \cos 2(\varphi_2 - \alpha)\} = \tfrac{1}{2}\{2x^2 - 1 + x\sqrt{4x^2 - 3}\},$$
$$\cos^2(\varphi_2 - \alpha) = \tfrac{1}{2}\{1 + \cos 2(\varphi_2 - \alpha)\} = \tfrac{1}{2}\{3 - 2x^2 - x\sqrt{4x^2 - 3}\}.$$

After extended reduction the above inequality becomes

$$(1 - x)^2(1 + x)^3(16x^5 + 16x^4 - 20x^3 - 20x^2 - 42x + 49) < 0.$$

In the range between 0 and 1, this inequality is satisfied by values of $x > 0.91401$ whence 2α or the vertical angle of the cone must be less than 23° 56′ for this maximum to exceed the area of the first surface.

To find the angle at which the delivery will be a maximum, assuming that the surface of the water remains level we must first find when the product of the area of the surface and the vertical velocity of the lip relative to the vertex is a maximum. If ω is the angular velocity of the cone, the velocity of the point A (relative to O) is $\omega\sqrt{h^2 + r^2}$ perpendicular to OA. Then there must be a maximum of

$$\frac{\pi r^2\cos\alpha\cos^{1/2}(\varphi + \alpha)\sqrt{h^2 + r^2}\sin(\varphi + \alpha)\omega}{\cos^{3/2}(\varphi - \alpha)}$$

or of $\{\cos(\varphi + \alpha)\sin^2(\varphi + \alpha)\}/\cos^3(\varphi - \alpha)$. This is a maximum when $3\cos(\varphi + \alpha)\cos 2\alpha = \cos(\varphi - \alpha)$ or

$$\tan\varphi = \cot\alpha(3\cos 2\alpha - 1)/(3\cos 2\alpha + 1).$$

This maximum will be greater than the initial rate of flow provided

$$\frac{\pi r^2 \cos\alpha \cos^{1/2}(\varphi+\alpha)\sqrt{h^2+r^2}\sin(\varphi+\alpha)}{\cos^{3/2}(\varphi-\alpha)} > \pi r^2\sqrt{h^2+r^2}\sin\alpha$$

which reduces for the above value of φ to

$$\left(\frac{3\cos 2\alpha - 1}{3\cos 2\alpha + 1}\right)^2 \cos^2 2\alpha(4 - 3\cos 2\alpha) > 0,$$

a condition that is true for all values of α, so that the maximum delivery is given in any case by

$$\varphi = \arctan\left(\frac{3\cos 2\alpha - 1}{3\cos 2\alpha + 1}\cot\alpha\right).$$

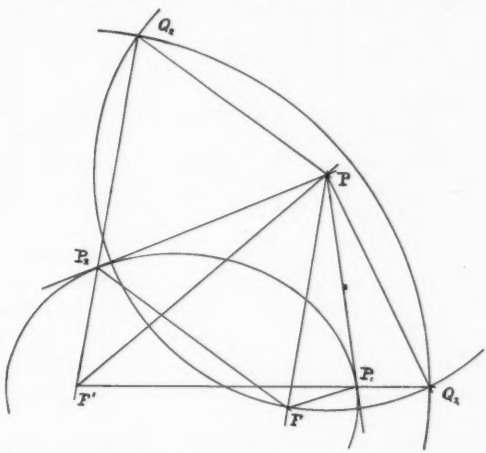
Also solved by E. M. BERRY, PHILIP FITCH and WILLIAM HOOVER.

2971 [1922, 179]. Proposed by C. E. HORNE, University of Porto Rico, Mayaguez, P. R.

Prove that the two tangents drawn to an ellipse from any external point subtend equal angles at a focus. (This problem is found in some text-books, but the proposer is anxious to see an analytic solution of it.)

I. SOLUTION BY THE LATE T. M. BLAKSLEE, Ames, Iowa.

The theorem will be proved by a construction for the two tangents from an external point P to the ellipse whose two foci are F' and F. Describe a circle with center P and radius PF, then a second circle with center F' and radius $2a$, the length of the major axis, cutting the first circle in Q_1 and Q_2. Let the straight lines Q_1F' and Q_2F' cut the ellipse in P_1 and P_2, respectively. Then $FP_2 = P_2Q_2$, since the sum of the focal radii, $F'P_2 + FP_2$, is equal to $2a$, the radius of the F' circle. Also $PF = PQ_2$ and hence $\angle PP_2Q_2 = \angle PP_2F$. It now follows from the theory of conic sections that PP_2 is the tangent at P_2. Similarly, PP_1 is the tangent at P_1.



Since the two triangles $Q_2F'P$ and $Q_1F'P$ are equal, the angles $\angle P_2F'P$ and $\angle P_1F'P$ are equal. A similar proof applies to the focus F.

NOTE BY THE EDITORS: The two circles always intersect in two points, for, since P is an external point, we have

$$|F'P - FP| \leqq F'F < 2a < F'P + FP.$$

An examination of the three pairs of congruent triangles arising in this construction shows that $\angle P_2PP_1$ and $\angle F'PF$ have a common bisector, observing that in the case of the ellipse the first angle encloses the second. This is a known theorem from which may be easily deduced the results given in C. Smith's *Conic Sections*, §§ 227, 228, where the above two theorems and others are proved.

Grace and Rosenberg (*The Conic*, p. 251) arrive at a neat solution by the use of polar coordinates. The tangents to the conic, $l = r + re\cos\theta$, at the points whose vectorial angles are α and β are

$$l = re\cos\theta + r\cos(\theta - \alpha)$$

and

$$l = re\cos\theta + r\cos(\theta - \beta).$$

Solving these for the r and θ of the point of intersection of the tangents, we find readily that $2\theta = \alpha + \beta$. The theorem follows.

II. Solution by A. M. Harding, University of Arkansas.

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be any two points on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$. The tangents at the two points have the equations

$$b^2x_1x + a^2y_1y = a^2b^2, \qquad b^2x_2x + a^2y_2y = a^2b^2 \tag{1}$$

and they intersect in the point P_3, whose coördinates are obtained by solving the pair of equations (1),

$$x_3 = \frac{a^2(y_2 - y_1)}{x_1y_2 - x_2y_1}, \qquad y_3 = \frac{b^2(x_1 - x_2)}{x_1y_2 - x_2y_1}. \tag{2}$$

The equation of the straight line through P_1 and the focus $F(ae, 0)$ is

$$y_1x - (x_1 - ae)y - aey_1 = 0; \tag{3}$$

and if this equation be divided by $\sqrt{y_1^2 + (x_1 - ae)^2} = a - ex_1$ we obtain the normal form. The distance, D, of P_3 from the line FP_1 is obtained by inserting in the equation (3), in the normal form, the coördinates of P_3 given in (2). Making this substitution and noting that $b^2x_1^2 + a^2y_1^2 = a^2b^2$, $a^2e^2 = a^2 - b^2$, the factor $a - ex_1$ divides out from numerator and denominator, and there results

$$D = \frac{b^2x_1x_2 + a^2y_1y_2 - a^2b^2}{\pm a(x_1y_2 - x_2y_1)}.$$

The distance of P_3 from FP_2 may now be obtained by interchanging the subscripts 1 and 2; but this does not alter the numerical value of D. Hence the angle $\angle P_2FP_1$ is bisected by FP_3.

Also solved by R. E. Gaines, William Hoover, J. B. Reynolds, W. J. Patterson, Hazel E. Schoonmaker and Mabel M. Young. Professor Blakslee sent in three other solutions.

2973 [1922, 225]. Proposed by N. K. Chaffee, Rutland, Vt.

A straight brass bar 800 feet long expands 8 inches. The ends are fixed, so that it is distorted. If the new shape is that of an arc of a circle of which the original bar is the chord, how far above the center of the bar in its original position will the center be in the new position?

It is instructive to guess at this distance before attempting to solve the problem.

Solution by A. Pelletier, Montreal, Canada.

Let 2θ be the central angle in radians subtended by the arc of $800\frac{2}{3}$ ft. and r the radius of the circle. From the equations $r \sin\theta = 400$ and $r\theta = 400\frac{1}{3}$ follow

$$\frac{\sin\theta}{\theta} = \frac{1200}{1201},$$

and this equation is to be solved for the positive root different from zero. Replacing $\sin\theta$ by its approximate value $\theta - \theta^3/6$, we obtain $\theta = \sqrt{6/1201} = .07068$. The required height is then $400 \tan\theta/2$, or with sufficient accuracy, $400\,\theta/2 = 14.14$ ft.

Also solved by A. Bogard, A. M. Harding and G. R. Livingston.

2974 [1922, 225]. Proposed by the late L. G. Weld.

A man standing on a straight railway track watches a train starting from a station half a mile distant and notices that it approaches at such speed that each puff of the exhaust is heard at the same instant that the next succeeding puff is seen. How long will it be before the train reaches him, the drive wheels of the locomotive being sixteen feet in circumference?

Solution by B. F. Finkel, Drury College.

Let c be the circumference of the drive wheel; n the number of puffs of the exhaust to each revolution of the drive wheel; d the distance from the station to the observer; and v, the velocity of sound.

Neglecting the velocity of light and the personal equation of the observer, we have the time, t_1, between the first and second puffs $= \frac{d}{v_s}$; t_2, the time between the second and third puffs $= \frac{d - \frac{c}{n}}{v_s}$; t_3, the time between the third and fourth puffs $= \frac{d - 2\frac{c}{n}}{v_s}$; $\cdots$; and, the time between the last puff and the last puff but one $= \frac{d - \left(\frac{nd}{c} - 1\right)\frac{c}{n}}{v_s}$, $c \div n$ being the distance traveled by the train between two consecutive puffs of the exhaust and $d/(c/n) = dn/c$ being the number of (c/n) intervals between the station and the observer.

Hence,

$$t = \sum_{i=1}^{nd/c} t_i = \frac{d}{v_s} + \frac{d - \frac{c}{n}}{v_s} + \frac{d - 2\frac{c}{n}}{v_s} + \cdots + \frac{d - \left(\frac{nd}{c} - 1\right)\frac{c}{n}}{v_s}$$

$$= \frac{d}{v_s} \cdot \frac{nd}{c} - \left[\frac{c}{nv_s} + \frac{2c}{nv_s} + \cdots + \frac{\left(\frac{nd}{c} - 1\right)\frac{c}{n}}{v_s}\right] = \frac{nd^2}{cv_s} - \frac{1}{2}\left[\frac{c}{nv_s} + \frac{\left(\frac{nd}{c} - 1\right)\frac{c}{n}}{v_s}\right]\left(\frac{nd}{c} - 1\right)$$

$$= \frac{nd^2}{2cv_s} + \frac{d}{2v_s} = \frac{d}{2v_s}\left(\frac{nd}{c} + 1\right).$$

In the modern locomotive engine, there are two cylinders, one on either side of the boiler, the difference of phase between the two being $\frac{1}{4}$. Hence, there are four puffs to each revolution of the drive-wheel.

Hence, substituting for c, 16; for n, 4; for d, 2640; and v_s, 1100, the velocity of sound at about 4.7° C., we find $t = 13$ min. 13.2 sec.

Also solved by H. A. BENDER, MOE BUCHMAN, S. A. COREY, and A. PELLETIER.

2979 [1922, 271]. Proposed by V. M. SPUNAR, Chicago, Illinois.

If a tangent be drawn from a variable point of an ellipse of length equal to n times the semi-conjugate diameter at the point, the locus of its extremity will be a concentric ellipse with semi-axes equal to $a\sqrt{n^2+1}$, $b\sqrt{n^2+1}$.

I. SOLUTION BY R. E. GAINES, University of Richmond.

Let $P_1(a\cos\varphi, b\sin\varphi)$ be the variable point on the ellipse; then the coördinates of an extremity P_2 of the diameter conjugate to OP_1 are $(a\sin\varphi, -b\cos\varphi)$. If OQ_2 is the prolongation of OP_2 to n times its length, the coördinates of Q_2 are $(na\sin\varphi, -nb\cos\varphi)$; and, if Q is the corresponding point on the locus, OQ_2QP_1 is a parallelogram and hence the coördinates of Q (α, β) are the sums of the coördinates of P_1 and Q_2. Hence

$$\frac{\alpha}{a} = \cos\varphi + n\sin\varphi, \qquad \frac{\beta}{b} = \sin\varphi - n\cos\varphi,$$

and, after squaring and adding, we have

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1 + n^2.$$

II. SOLUTION BY A. M. HARDING, University of Arkansas.

Let $P_1(x_1, y_1)$ be the variable point on the ellipse whose equation is $b^2x^2 + a^2y^2 = a^2b^2$. The coördinates of the extremity of the conjugate diameter are $(ay_1/b, -bx_1/a)$. We then have from the conditions of the problem

$$b^2x_1^2 + a^2y_1^2 = a^2b^2, \tag{1}$$

$$b^2x_1\alpha + a^2y_1\beta = a^2b^2, \tag{2}$$

$$(\alpha - x_1)^2 + (\beta - y_1)^2 = n^2\left[\frac{a^2y_1^2}{b^2} + \frac{b^2x_1^2}{a^2}\right], \tag{3}$$

where (α, β) is the extremity of the tangent.

From (1) and (2) we find

$$b^2x_1(\alpha - x_1) + a^2y_1(\beta - y_1) = 0,$$

or

$$\frac{\alpha - x_1}{\beta - y_1} = -\frac{a^2y_1}{b^2x_1}. \tag{4}$$

Then from (3) and (4),

$$b(\alpha - x_1) = nay_1,$$
$$a(\beta - y_1) = -nbx_1.$$

Whence,

$$bx_1(1 + n^2) = b\alpha - na\beta,$$
$$ay_1(1 + n^2) = a\beta + nb\alpha.$$

Square and add,

$$a^2b^2(1 + n^2)^2 = (b^2\alpha^2 + a^2\beta^2)(1 + n^2),$$

or

$$b^2\alpha^2 + a^2\beta^2 = (1 + n^2)a^2b^2.$$

Whence,

$$\frac{\alpha^2}{a^2(1 + n^2)} + \frac{\beta^2}{b^2(1 + n^2)} = 1.$$

The locus is, therefore, a concentric ellipse whose semi-axes are $a\sqrt{1 + n^2}$, $b\sqrt{1 + n^2}$. It may be easily shown that a similar theorem holds true for the hyperbola.

Also solved by T. M. BLAKSLEE, E. J. OGLESBY, and J. B. REYNOLDS.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

The principal features of the annual meeting of the Inland Empire Council of Teachers of Mathematics were addresses by Professor W. A. BRATTON, of Whitman College, on "What a high school teacher of mathematics ought to know about Einstein," and by Professor F. L. GRIFFIN, of Reed College, on "The rôle of mathematics in human progress." Professor W. C. Eells, of Whitman College, has retired as president of the Council after three years' service. Professor C. A. Isaacs, of Washington State College, is a member of the executive committee.

The Ohio Section of the Association has definitely interested itself in the high school situation through the formulation by a committee of a letter to high school seniors presenting in a telling form reasons for continuing the study of mathematics in college, and of a letter to high school freshmen setting forth in a form particularly well suited to pupils of that age the importance of following up the study of arithmetic by that of algebra and geometry. The State Department of Education of Ohio has agreed to send copies of these letters to every superintendent of schools in the state. This substantial piece of committee activity will commend itself to other sections of the Association as well as to other organizations interested in setting forth the value of mathematics.

Professor A. A. MICHELSON of the University of Chicago has been elected president of the National Academy of Sciences. Professors E. W. BROWN of Yale and D. L. WEBSTER of Stanford have been elected to membership in the Academy.

Dr. H. M. DADOURIAN, associate professor of physics at Trinity College, has been appointed Seabury professor of mathematics.

Dr. LOUIS WEISNER, of Columbia University, has been appointed instructor of mathematics at the University of Rochester.

Mr. J. B. ROSENBACH, instructor in mathematics at the Carnegie Institute of Technology, has been promoted to an assistant professorship of mathematics.

Mr. O. H. RECHARD, Jr., instructor in mathematics at the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Wyoming.

Professor C. N. MILLS, professor of mathematics at Heidelberg University, Tiffin, O., and one of the associate editors of the MONTHLY, has accepted a professorship at the South Dakota State Normal. He taught for six years at South Dakota State College before moving to Ohio.

Associate Professor R. C. ARCHIBALD of Brown University, past president of the Association and for several years editor-in-chief of the MONTHLY, has been promoted to a full professorship.

Chancellor A. B. CHACE of Brown University, as a vice-president, represented the Mathematical Association of America at the inauguration of Dr. S. W. Stratton as president of the Massachusetts Institute of Technology on June 11, 1923.

Assistant Professor W. L. CRUM of Yale University has been appointed assistant professor of statistics in the department of economics of Harvard University.

Dr. C. H. YEATON and Dr. F. E. CARR have been given permanent appointments as assistant professors of mathematics at Oberlin College.

Mr. C. A. GARABEDIAN has received from Harvard University the degree of doctor of philosophy and an appointment as Parker Travelling Fellow for the next academic year.

Mr. NORMAN ANNING, instructor in mathematics at the University of Michigan, has been promoted to an assistant professorship.

On commencement day, June 20, at Brown University the honorary degree of Sc.D. was conferred upon Professor G. D. BIRKHOFF of Harvard University.

THE CINCINNATI MEETING OF THE ASSOCIATION

The Mathematical Association of America will meet at the University of Cincinnati, in affiliation with the American Association for the Advancement of Science and the Chicago Section of the American Mathematical Society, on Thursday and Friday, December 27–28, 1923. The first session of the Mathematical Association will be held on Thursday afternoon, a joint session of the Association with Section A of the American Association and the Chicago Section will be held on Friday afternoon, and another session of the Association simultaneous with that of the Chicago Section will be held on Friday morning. Other sessions of the Chicago Section will occur on Saturday morning and probably Saturday afternoon. This plan will enable our members to attend the mathematical meetings during these three days and then have free opportunity to visit the meetings of the American Association during the first three days of the next week.

NOTABLE GIFTS TO THE ASSOCIATION

A notable bequest in the form of a last will and testament has just been executed in favor of the Mathematical Association of America which is designated as sole legatee. The name of the donor is withheld at present by request, but the person through whose influence this bequest was secured is Mr. C. C. Carter of Bluffs, Illinois, who has been a member of the Association since 1918. He deserves great credit for the service which he has thus rendered to the Association, a kind of service which may lie within reach of any member who is on the alert for such an opportunity. To quote his own modest words: "I am not much of a mathematician, but I have all my life studied mathematics as a pastime and I have always had a responsive chord in my cardiac impulses for the fellow who sits up nights to pore over the cabalistic characters of this great subject. While this is only a will, I have perfect confidence that its provisions will be faithfully carried out. I am thoroughly familiar with the property, which is mostly in land, and of course at low ebb in the market just now, but I conservatively estimate its potential value at $100,000."

The will provides for the creation of a Memorial Fund whose income "shall be used for the promotion of mathematical science and its teaching in America in such manner as the officers of the above named corporation may deem fit." The Trustees of the Association are thus given a free hand in determining the objects to which the funds should be applied, the only condition being that "each and every thing done by the financial aid of the above mentioned fund shall bear some distinguishing mark, print, character, or notification, showing the source from which the financial aid to do the said thing or act was derived."

Another notable service to the Association has been very unostentatiously rendered by Chancellor A. B. Chace, of Brown University, who is a Vice-President of the Association for 1923. For each of the two years, 1922, 1923, he has contributed the sum of four hundred dollars in cash to the current funds of the Association. To these generous gifts is largely due the fact that the Association has been able to proceed without drawing upon its slender reserve funds inherited from the former management of the MONTHLY. Chancellor Chace is not engaged professionally in the field of mathematics, but he evidently finds great delight and satisfaction in certain phases of mathematical activity. For instance, he has been occupied for some years in making a translation of the Ahmes Papyrus which he expects soon to publish. The Association is honored by the presence of such a person on its membership roll and among its officers.

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